Bi-conformal vector fields and their applications

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Abstract. We introduce the concept of *bi-conformal transformation*, as a generalization of conformal ones, by allowing two orthogonal parts of a manifold with metric $\mathbf{g}$ to be scaled by different conformal factors. In particular, we study their infinitesimal version, called bi-conformal vector fields. We show that these are characterized by the differential conditions $\pounds_{\vec{\xi}}\,\mathbf{P} \propto \mathbf{P}$ and $\pounds_{\vec{\xi}}\,\mathbf{\Pi} \propto \mathbf{\Pi}$ where $\mathbf{P}$ and $\mathbf{\Pi}$ are orthogonal projectors ($\mathbf{P}+\mathbf{\Pi}=\mathbf{g}$). Keeping $\mathbf{P}$ and $\mathbf{\Pi}$ fixed, the set of bi-conformal vector fields is a Lie algebra which can be finite or infinite dimensional according to the dimensionality of the projectors. We determine (i) when an infinite-dimensional case is feasible and its properties, and (ii) a *normal system* for the generators in the finite-dimensional case. Its integrability conditions are also analyzed, which in particular provides the maximum number of linearly independent solutions. We identify the corresponding maximal spaces, and show a necessary geometric condition for a metric tensor to be a double-twisted product. More general "breakable" spaces are briefly considered. Many known symmetries are included, such as conformal Killing vectors, Kerr-Schild vector fields, kinematic self-similarity, causal symmetries, and rigid motions.

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1. Introduction

Symmetry transformations have been a subject of research over the years. In General Relativity they have been used for different purposes, ranging from the classification of exact solutions of the field equations to the generation techniques for new solutions [39]. In this work, we are interested in the study of continuous transformation groups with certain properties acting on a metric manifold (see [25, 27] for a precise definition of this). A classification of the outstanding cases can be found in [30] where they are sorted according to the *differential conditions* complied by the infinitesimal generators. This condition involves the Lie derivative of the metric tensor or other geometric objects —such as the connection or the curvature tensor—. The symmetries classified in [30] have received a great deal of attention. However, as a matter of fact, it is difficult to find in the literature studies of symmetries characterized by other differential conditions. Some examples can be found in [16, 29, 41, 44, 23].

In this paper we will pursue this line of research and present a new type of group of transformations: those diffeomorphisms which scale two pieces of the metric tensor by unequal factors. We call them *bi-conformal transformations*. We will not restrict our presentation to four-dimensional spacetimes, so that our results will be valid in any n-dimensional differential manifold V with a smooth metric tensor $\mathbf{g}$ of any signature. Bi-conformal transformations can be univocally characterized by

a *symmetric square root* of $\mathbf{g}$ (see next section for its definition; this is a Lorentz tensor in Lorentzian signature, see [6, 24]), or equivalently by two complementary orthogonal projectors. The infinitesimal generators are well defined and we present the necessary and sufficient differential conditions they fulfill, which involve the Lie derivatives of the metric tensor and of the square root, or equivalently of the two projectors. This differential condition can be understood as stating that the generating vectors, called *bi-conformal vector fields*, are generalized conformal motions for both projectors. The properties of bi-conformal vector fields are studied, and we show that they constitute a Lie algebra which can be finite or infinite dimensional. We identify the cases where the latter case may happen. We also prove that, in the former case, a normal system—for $p, n-p \neq 1, 2$ where p is the trace of one of the projectors—can be achieved, so that the integrability conditions and the maximum number of linearly independent bi-conformal vector fields are found. This turns out to be $(p+1)(p+2)/2+(n-p+1)(n-p+2)/2$. We show that this maximum number is attained in double-twisted product spaces with flat leaves (i.e. a metric breakable in two conformally flat pieces where the conformal factor of each part depend on all the coordinates of the manifold). We also find a necessary geometric condition for a space to admit such form in local coordinates.

The outline of the paper is as follows: in section 2 we set the notation and study the basic properties of square roots. Bi-conformal vector fields are introduced in section 3 whereas in section 4 we study groups of bi-conformal transformations and give the general form taken by the metric tensor in a coordinate system adapted to the symmetry. The Lie algebra of bi-conformal vector fields is the subject of section 5. The normal system for the finite-dimensional case is obtained in section 6 together with the highest dimension of the Lie algebra. Part of the integrability conditions of the afore-mentioned equations are considered in section 7. Finally explicit examples of bi-conformal vector fields are presented in section 8.

2. Preliminaries.

We start by setting the notation and conventions to be used in this work. $(V, \mathbf{g})$ will stand for a smooth n-dimensional manifold with metric $\mathbf{g}$. The metric signature is arbitrary although in some of our results we will specialize $\mathbf{g}$ to a Lorentzian metric (signature convention $(+,-,\ldots,-)$) in order to highlight the applications to n-dimensional spacetimes. Latin characters running from 1 to n will be used for tensor indexes. Vectors and contravariant (covariant) tensors will be denoted with arrowed (un-arrowed) boldface characters whenever they are expressed in index-free notation. As usual contravariant and covariant tensors are related by the rule of raising and lowering of indexes. Thus if $\vec{\mathbf{T}}$ is a rank-r contravariant tensor we will use the same un-arrowed symbol $\mathbf{T}$ for the tensor obtained by lowering all the indexes. Index notation though will be used in most of the paper for tensors. Round and square brackets enclosing indexes will stand for symmetrization and anti-symmetrization respectively. We review next very briefly some basic geometric concepts in order to show the notation we use. In $(V, \mathbf{g})$ we define the tangent space $T_x(V)$ at a point $x \in V$, the tangent and cotangent bundles $T(V)$, $T^*(V)$ and the bundle $T^r_s(V)$ of r-covariant s-contravariant tensors in the usual way. All differentiable sections of the bundle (vector fields) $T(V)$ and other tensor bundles will be assumed smooth. We will use the same notation for sections as for vectors and tensors unless the context requires to use different notations.

Recall that any C^1 vector field $\vec{\boldsymbol{\xi}}$ on a differentiable manifold defines a local group of *local* diffeomorphisms $\{\varphi_s\}$ that is to say, each member φ_s of the family is a local diffeomorphism and φ_0 is the identity on V. In addition to this $(\varphi_{s_1} \circ \varphi_{s_2})(x) = \varphi_{s_1+s_2}(x)$ holds whenever all the objects appearing in both sides of the equality make sense. In local coordinates $\{x^a\}$ we have the correspondence

$$\xi^a(\varphi_s(x)) = \frac{d\varphi_s^a(x)}{ds}, \qquad \varphi_0(x) = x$$

which relates the vector field $\vec{\boldsymbol{\xi}}$ with its generated local group of local diffeomorphisms by means of standard theorems on differential equations. When this local group is formed by global diffeomorphisms $\varphi_s : V \to V$, $s \in I$ then $\vec{\boldsymbol{\xi}}$ is called a *complete* vector field being I an interval of the real line containing 0. For a Hausdorff manifold this is equivalent to $I = \mathbb{R}$ or in other words the integral curves of this vector field can be extended to every value of their parameter. As is well known vector fields can be regarded as differential operators acting on the set of smooth functions of the manifold V and in this picture the Lie bracket of two C^1 vector fields $\vec{\boldsymbol{\xi}}_1$, $\vec{\boldsymbol{\xi}}_2$ is

$$[\vec{\boldsymbol{\xi}}_1, \vec{\boldsymbol{\xi}}_2](f) \equiv \vec{\boldsymbol{\xi}}_1(\vec{\boldsymbol{\xi}}_2(f)) - \vec{\boldsymbol{\xi}}_2(\vec{\boldsymbol{\xi}}_1(f)).$$

The set of smooth vector fields generates an infinite dimensional Lie algebra by means of the Lie bracket operation. This is usually denoted by $\mathfrak{X}(V)$ and not all the elements of $\mathfrak{X}(V)$ are complete unless the dimension of the Lie algebra is finite. We can use the (local) one-parameter group of diffeomorphisms $\{\varphi_s\}$ generated by $\vec{\boldsymbol{\xi}}$ to define, for any $\vec{\mathbf{T}} \in T^r(V)$, the family of tensor fields $\varphi'_s\vec{\mathbf{T}}$ where φ'_s is the push-forward of φ_s. There is an obvious counterpart for covariant tensor fields $\mathbf{T}$ using the pull-back φ^*_s. The Lie derivative of $\vec{\mathbf{T}}$ ($\mathbf{T}$) is another tensor field of the same rank defined by

$$\pounds_{\vec{\boldsymbol{\xi}}}\vec{\mathbf{T}} = \lim_{s\to 0}\frac{\varphi'_{-s}\vec{\mathbf{T}} - \vec{\mathbf{T}}}{s}, \quad \pounds_{\vec{\boldsymbol{\xi}}}\mathbf{T} = \lim_{s\to 0}\frac{\varphi^*_s\mathbf{T} - \mathbf{T}}{s}.$$

All these geometric definitions are well known and can be found in many text books (see e. g. [14]).

2.1. Square roots.

We present next a concept which will play a very important role in this work.

Definition 2.1 *Let $\mathbf{g}|_x$ be a nondegenerate bilinear form defined on $T_x(V)$. A symmetric tensor S_{ab} on $T_x(V)$ is called a square root of $\mathbf{g}|_x$ if*

$$S_{ap}S^p{}_b = \mathrm{g}_{ab}.$$

If g_{ab} has Lorentzian signature then the tensor S_{ab} is called a *Lorentz tensor*. Clearly Lorentz tensors with an index raised are involutory Lorentz transformations (a linear transformation is said to be involutory if its inverse is the transformation itself) but they can also be characterized as *superenergy tensors* of certain normalized simple forms as it was shown in [6]:

Proposition 2.1 *Every Lorentz tensor S_{ab} is proportional to the superenergy tensor of a simple form $\boldsymbol{\Omega}$.*

Without going into further details which are beyond the scope of this paper, we will just write down the definition of the superenergy tensor $T_{ab}\{\mathbf{\Omega}\}$ of a p-form $\mathbf{\Omega}$ [36, 6]

$$T_{ab}\{\mathbf{\Omega}\} = \frac{(-1)^{p-1}}{(p-1)!}\left(\Omega_{ac_2\dots c_p}\Omega_b{}^{c_2\dots c_p} - \frac{1}{2p}\mathrm{g}_{ab}\Omega_{c_1\dots c_p}\Omega^{c_1\dots c_p}\right). \quad (2.1)$$

The form $\mathbf{\Omega}$ is said to be simple if it can be decomposed as the wedge product of 1-forms and the normalization required is

$$\mathbf{\Omega}\cdot\mathbf{\Omega} \equiv \Omega_{c_1\dots c_p}\Omega^{c_1\dots c_p} = 2p!(-1)^{p-1}. \quad (2.2)$$

Denoting by $\boldsymbol{k}_1,\dots,\boldsymbol{k}_p$ a set of 1-forms such that $\mathbf{\Omega} = \boldsymbol{k}_1\wedge\dots\wedge\boldsymbol{k}_p$ we have that $Span\{\vec{\mathbf{k}}_1,\dots,\vec{\mathbf{k}}_p\}$ and $\perp Span\{\vec{\mathbf{k}}_1\dots,\vec{\mathbf{k}}_p\}$ are the only eigenspaces of $S^a{}_b$ with corresponding eigenvalues $+1$ and -1 respectively. Another important property is the invariance $T_{ab}\{\mathbf{\Omega}\} = T_{ab}\{\pm *\mathbf{\Omega}\}$ where $*\mathbf{\Omega}$ is the hodge dual of $\mathbf{\Omega}$. Thus, whenever we speak of the Lorentz tensor of a p-form this must be understood up to duality and sign. With the normalization chosen above, $Span\{\vec{\mathbf{k}}_1,\dots,\vec{\mathbf{k}}_p\}$ is a timelike subspace ($\mathbf{\Omega}$ is "timelike") and $\perp Span\{\vec{\mathbf{k}}_1,\dots,\vec{\mathbf{k}}_p\}$ is its spacelike complement ($*\mathbf{\Omega}$ is spacelike). Some of these results can be translated to arbitrary square roots.

Proposition 2.2 *Every square root $S^a{}_b$, regarded as an endomorphism, has $+1$ and -1 as unique eigenvalues. Furthermore $S^a{}_b$ is diagonalizable so the space $T_x(V)$ decomposes in a direct sum of the two eigenspaces associated to the eigenvalues $+1$ and -1.*

Proof : The first statement of the proof is trivial. To prove the second one we must note that the endomorphisms $P^a_b = (\delta^a{}_b + S^a{}_b)/2$, $\Pi^a{}_b = (\delta^a{}_b - S^a{}_b)/2$ are idempotent, their composition in any order gives the zero endomorphism, both have vanishing determinant and $\delta^a{}_b = P^a_b + \Pi^a{}_b$. From this we deduce that the direct sum of the eigenspaces of $S^a{}_b$, with eigenvalues $+1$ and -1, is the total space or in other words $S^a{}_b$ is diagonalizable. □

The following straightforward corollary is deduced from this proposition.

Corollary 2.1 *The tensors*

$$P_{ab} = (\mathrm{g}_{ab} + S_{ab})/2, \quad \Pi_{ab} = (\mathrm{g}_{ab} - S_{ab})/2 \quad (2.3)$$

constructed from any square root S_{ab} are orthogonal projectors on the eigenspaces of eigenvalue $+1$ and -1 respectively.

For the case of Lorentzian signature the subspaces onto which P_{ab} and Π_{ab} project coincide with $Span\{\mu(\mathbf{S})\}$ and $\perp Span\{\mu(\mathbf{S})\}$ where $\mu(\mathbf{S})$ is the set of null vectors $\vec{\mathbf{k}}$ such that $\mathbf{S}(\vec{\mathbf{k}},\vec{\mathbf{k}}) = 0$ (see proposition A.3 of [24] for further details).

By means of previous results, we can now generalize proposition 2.1 to cases in which the metric tensor is not necessarily Lorentzian.

Theorem 2.1 *Any square root S_{ab} can be written, up to sign, as in (2.1) for a simple form $\mathbf{\Omega}$ normalized, up to sign, as in (2.2).*

Proof : Under the assumptions of this theorem a not very long calculation shows that for a p-form $\Omega_{a_1\dots a_p}$ we have

$$T_{ap}\{\mathbf{\Omega}\}T^p_b\{\mathbf{\Omega}\} = \frac{(\mathbf{\Omega}\cdot\mathbf{\Omega})^2}{(2p!)^2},$$

(the procedure is the same as for the Lorentzian case, see the proof of proposition 3.2 in [6]). Thus formula (2.1) still defines a square root if $\mathbf{\Omega}\cdot\mathbf{\Omega}=\pm 2p!$ in the case of a non-Lorentzian metric. Conversely if $S^a{}_b$ is a square root then proposition 2.2 tells us that we can write $T_x(V)$ as a direct sum of the eigenspaces of $S^a{}_b$ with eigenvalues +1 and -1 denoted by V_+ and V_- respectively. Furthermore $S^a{}_b$ is a symmetric endomorphism with respect to the metric $\mathbf{g}|_x$ so $V_+^\perp = V_-$. Now choose a basis $\{\vec{\mathbf{k}}_1,\ldots,\vec{\mathbf{k}}_p\}$ of V_+ (we take it of dimension p) and calculate the tensor $\mathbf{T}\{\mathbf{\Sigma}\}$ with $\mathbf{\Sigma}=\boldsymbol{k}_1\wedge\ldots\wedge\boldsymbol{k}_p$. Use of formula (2.1) implies that $\vec{\mathbf{k}}_j$ $j=1,\ldots p$, are eigenvectors of $T^a{}_b\{\mathbf{\Sigma}\}$ with eigenvalue $(-1)^{p-1}\mathbf{\Sigma}\cdot\mathbf{\Sigma}/(2p!)$ and the same happens for any vector in V_- (but now the eigenvalue has the opposite sign). The identity

$$\frac{1}{(p-1)!}\Sigma^{ab_1\ldots b_{p-1}}\Sigma_{bb_1\ldots b_{p-1}}+\frac{|\det(\mathbf{g})|}{\det(\mathbf{g})(n-p-1)!}(*\Sigma)^{ab_{p+2}\ldots b_n}(*\Sigma)_{bb_{p+2}\ldots b_n}=\frac{\mathbf{\Sigma}\cdot\mathbf{\Sigma}}{p!}\delta^a{}_b,$$

must be used along the way to prove this. Hence the tensor $T^a{}_b\{\mathbf{\Sigma}\}$ has the same spectral properties as $S^a{}_b$ which is only possible if both are proportional. The proportionality factor can be fixed to +1 or −1 by choosing the normalization $\mathbf{\Sigma}\cdot\mathbf{\Sigma}=\pm 2p!$. □

Remark. Note that the sign of $\mathbf{\Sigma}\cdot\mathbf{\Sigma}$ is fixed and characteristic to the subspace V_+ (if the metric has Lorentzian signature this sign allows us to decide the causal character of the subspace V_+). One can thus choose the sign in the normalization condition $\mathbf{\Sigma}\cdot\mathbf{\Sigma}=\pm 2p!$ so that $\mathbf{T}\{\mathbf{\Sigma}\}=\mathbf{S}$.

3. Bi-conformal vector fields

Definition 3.1 $\vec{\boldsymbol{\xi}}$ *is said to be a* bi-conformal vector field *if it fulfills the differential conditions*

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}=\alpha\mathbf{g}+\beta\mathbf{S},\quad \pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{S}=\alpha\mathbf{S}+\beta\mathbf{g},\tag{3.1}$$

where $\mathbf{S}$ *is a symmetric square root of* $\mathbf{g}$ *and* α, β *smooth functions.*

Following [16] the functions α and β will be called *gauges* of the symmetry and their true relevance will become clear later. Suffice it to say here they play a role analogous to the factor ψ appearing in the differential condition $\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}=2\psi\mathbf{g}$ satisfied by conformal motions [43].

If the signature admits null vectors then there is a variant of previous definition known as generalized Kerr Schild vector fields where S_{ab} is no longer a square root but it takes the form $S_{ab}=k_ak_b$ with $k^ak_a=0$ (this can be also characterized as $S_{ap}S^p{}_b=0$). The explicit differential condition is now

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}=\alpha\mathbf{g}+\beta\boldsymbol{k}\otimes\boldsymbol{k},\quad \pounds_{\vec{\boldsymbol{\xi}}}\,\boldsymbol{k}=\gamma\boldsymbol{k}.\tag{3.2}$$

Generalized Kerr-Schild vector fields are a generalization of Kerr-Schild vector fields studied in [16] given by (3.2) with $\alpha=0$. Both bi-conformal vector fields and generalized Kerr-Schild vector fields can be written in a gauge-free way as we show in the next theorem.

Theorem 3.1 *A vector field* $\vec{\boldsymbol{\xi}}$ *is either a bi-conformal vector field or a generalized Kerr-Schild vector field if and only if*

$$(\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}\times\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g})\wedge\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}\wedge\mathbf{g}=0,\quad (\pounds_{\vec{\boldsymbol{\xi}}}\,\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g})\wedge\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}\wedge\mathbf{g}=0.\tag{3.3}$$

Proof : The inner product $\times$ of two rank-2 tensors T_{ab} and M_{ab} is defined by $(T \times M)_{\ ab} = T_{ac}M^{c}{}_{b}$ and their wedge product $\wedge$ is the typical wedge product in $T_2^0(V)$ considered as a vector space, e.g. $(T \wedge M)_{abcd} = T_{ab}M_{cd} - T_{cd}M_{ab}$. It is clear that the statement of this theorem is equivalent to demanding the existence of functions λ_1, λ_2, λ_3 and λ_4 with the properties

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g} \times \pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g} = \lambda_1 \pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g} + \lambda_2 \mathbf{g}, \quad \pounds_{\vec{\boldsymbol{\xi}}}\,\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g} = \lambda_3 \mathbf{g} + \lambda_4 \pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g}, \tag{3.4}$$

so we will prove the equivalence of these equations to those characterizing bi-conformal vector fields and generalized Kerr-Schild vector fields. Straightforward calculations show that the above expressions are fulfilled with

$$\lambda_1 = 2\alpha,\ \lambda_2 = \beta^2 - \alpha^2,\ \lambda_3 = -\alpha^2 + \beta^2 + \pounds_{\vec{\boldsymbol{\xi}}}\,\alpha - \frac{\alpha}{\beta}\pounds_{\vec{\boldsymbol{\xi}}}\,\beta,\ \lambda_4 = \frac{1}{\beta}\pounds_{\vec{\boldsymbol{\xi}}}\,\beta + 2\alpha,$$

if $\vec{\boldsymbol{\xi}}$ is a bi-conformal vector field, and with

$$\lambda_1 = 2\alpha,\ \lambda_2 = -\alpha^2,\ \lambda_3 = \pounds_{\vec{\boldsymbol{\xi}}}\,\alpha - \frac{\alpha}{\beta}\pounds_{\vec{\boldsymbol{\xi}}}\,\beta - 2\gamma\alpha,\ \lambda_4 = \frac{1}{\beta}\pounds_{\vec{\boldsymbol{\xi}}}\,\beta + \alpha + 2\gamma,$$

for a generalized Kerr-Schild vector field. Conversely, assuming that (3.4) holds and setting $\mathbf{T} = \pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g} - \frac{1}{2}\lambda_1\mathbf{g}$, the first of these equations implies that $\mathbf{T} \times \mathbf{T} \propto \mathbf{g}$ so that $\mathbf{T}$ is proportional to a square root of $\mathbf{g}$ or $\mathbf{T} \times \mathbf{T} = 0$. In the former case, $\mathbf{T} = \beta\mathbf{S}$ for some square root $\mathbf{S}$ whence $\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{g} = \alpha\mathbf{g} + \beta\mathbf{S}$ with $\alpha = \frac{1}{2}\lambda_1$ whose substitution in the second equation of (3.4) allows us to get

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\mathbf{S} = \mu_1\mathbf{g} + \mu_2\mathbf{S},\quad \mu_1 = \frac{1}{\beta}(\lambda_3 + \lambda_4\alpha - \pounds_{\vec{\boldsymbol{\xi}}}\,\alpha - \alpha^2),\quad \mu_2 = \lambda_4 - \frac{1}{\beta}\pounds_{\vec{\boldsymbol{\xi}}}\,\beta - \alpha.$$

The functions μ_1 and μ_2 can be further constrained by applying $\pounds_{\vec{\boldsymbol{\xi}}}$ to the relation $\delta^{a}{}_{b} = S^{a}{}_{c}S^{c}{}_{b}$ getting $\mu_1 = \beta$, $\mu_2 = \alpha$ from what we deduce that $\vec{\boldsymbol{\xi}}$ is a bi-conformal vector field. In the case $\mathbf{T} \times \mathbf{T} = 0$ then either $\mathbf{T} = 0$, so that $\vec{\boldsymbol{\xi}}$ is a conformal Killing vector, or $\mathbf{T} = \boldsymbol{k} \otimes \boldsymbol{k}$, with $\boldsymbol{k}$ null (this can only happen if the metric admits null vectors) so that

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\boldsymbol{k} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \pounds_{\vec{\boldsymbol{\xi}}}\,\boldsymbol{k} = \mu_1\mathbf{g} + \mu_2\boldsymbol{k} \otimes \boldsymbol{k},$$

which only makes sense if $\mu_1 = 0$. This gives the differential condition characterizing generalized Kerr-Schild vector fields at once. $\square$

Bi-conformal vector fields will be studied paying special attention to their geometrical meaning. To start with we show how the second equation of (3.1) can be rewritten in terms of the p-form giving rise to the square root $\mathbf{S}$ of $\mathbf{g}$ entering in the definition of bi-conformal vector fields.

Proposition 3.1 *The second equation of (3.1) admits the following equivalent forms*

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\boldsymbol{\Omega} = \frac{p}{2}(\alpha + \beta)\boldsymbol{\Omega}, \quad \pounds_{\vec{\boldsymbol{\xi}}}\,\vec{\boldsymbol{\Omega}} = -\frac{p}{2}(\alpha + \beta)\vec{\boldsymbol{\Omega}}, \tag{3.5}$$

where $\boldsymbol{\Omega}$ *is a simple* p*-form such that* $\mathbf{S} = \mathbf{T}\{\boldsymbol{\Omega}\}$ *and the first of (3.1) is assumed.*

Proof : From equations (3.1) it follows, by means of $\pounds_{\vec{\boldsymbol{\xi}}}\,\mathrm{g}^{ab} = -\alpha\mathrm{g}^{ab} - \beta S^{ab}$, that $\pounds_{\vec{\boldsymbol{\xi}}}\,S^{a}{}_{b} = 0$ so that for any vector $\vec{\mathbf{k}}$ intervening as a factor in the p-vector $\vec{\boldsymbol{\Omega}}$ we have $S^{a}{}_{p}\pounds_{\vec{\boldsymbol{\xi}}}\,k^{p} = \pounds_{\vec{\boldsymbol{\xi}}}\,k^{a}$ as follows by Lie derivating the equation $S^{a}{}_{p}k^{p} = k^{a}$. This means that $\pounds_{\vec{\boldsymbol{\xi}}}\,k^{a}$ is also an eigenvector of $S^{a}{}_{b}$ and thus $\pounds_{\vec{\boldsymbol{\xi}}}\,k^{a}$ is a linear combination of the

vectors which build up $\vec{\mathbf{\Omega}}$ whence $\pounds_{\vec{\boldsymbol{\xi}}}\vec{\mathbf{\Omega}} \propto \vec{\mathbf{\Omega}}$. Using the relation between $\mathbf{\Omega}$ and $\vec{\mathbf{\Omega}}$ and the first of (3.1) we get a similar formula for $\pounds_{\vec{\boldsymbol{\xi}}}\mathbf{\Omega}$. The proportionality factors are fixed by Lie derivating the normalization condition on $\mathbf{\Omega}\cdot\mathbf{\Omega}$ written before, arriving thus at (3.5). Conversely, if any of (3.5) together with the first equation of (3.1) hold we can work out the other equation of (3.5) using the relation between $\mathbf{\Omega}$ and $\vec{\mathbf{\Omega}}$ together with $\Omega_{a_1\dots a_p}S^{a_1}_{\ b_1} = \Omega_{b_1 a_2\dots a_p}$. Then $\pounds_{\vec{\boldsymbol{\xi}}} S_{ab}$ is calculated by just derivating (2.1) (as all the required Lie derivatives are known) getting the second equation of (3.1). □

4. Bi-conformal transformations

The structure of the differential conditions for bi-conformal vector fields allows us to find explicit expressions for $\varphi_s^*\mathbf{g}$ and $\varphi_s^*\mathbf{S}$. To that end we rewrite equations (3.1) as

$$\pounds_{\vec{\boldsymbol{\xi}}}(\mathbf{g}+\mathbf{S}) = (\alpha+\beta)(\mathbf{g}+\mathbf{S}), \quad \pounds_{\vec{\boldsymbol{\xi}}}(\mathbf{g}-\mathbf{S}) = (\alpha-\beta)(\mathbf{g}-\mathbf{S}). \tag{4.1}$$

Observe that these are differential conditions on the projectors P_{ab} and Π_{ab}, so that they are conformally invariant, and one obviously gets

$$\pounds_{\vec{\boldsymbol{\xi}}} P^a{}_b = \pounds_{\vec{\boldsymbol{\xi}}} \Pi^a{}_b = 0. \tag{4.2}$$

Now if we take into account the identity

$$\frac{d(\varphi_s^*\mathbf{T})}{ds} = \varphi_s^*(\pounds_{\vec{\boldsymbol{\xi}}}\mathbf{T}), \tag{4.3}$$

holding for every section $\mathbf{T}$ of $T_r^0(V)$, equations (4.1) can be integrated yielding

$$\varphi_s^*\mathbf{g} + \varphi_s^*\mathbf{S} = (\mathbf{g}+\mathbf{S})\exp\left[\int_0^s dt(\alpha(\varphi_t)+\beta(\varphi_t))\right] \tag{4.4}$$

$$\varphi_s^*\mathbf{g} - \varphi_s^*\mathbf{S} = (\mathbf{g}-\mathbf{S})\exp\left[\int_0^s dt(\alpha(\varphi_t)-\beta(\varphi_t))\right], \tag{4.5}$$

from what we get the formulae for $\varphi_s^*\mathbf{g}$ and $\varphi_s^*\mathbf{S}$

$$\varphi_s^*\mathbf{g} = \exp\left\{\int_0^s dt\ \alpha(\varphi_t)\right\}\left[\cosh\left(\int_0^s dt\ \beta(\varphi_t)\right)\mathbf{g} + \sinh\left(\int_0^s dt\ \beta(\varphi_t)\right)\mathbf{S}\right], \tag{4.6}$$

$$\varphi_s^*\mathbf{S} = \exp\left\{\int_0^s dt\ \alpha(\varphi_t)\right\}\left[\cosh\left(\int_0^s dt\ \beta(\varphi_t)\right)\mathbf{S} + \sinh\left(\int_0^s dt\ \beta(\varphi_t)\right)\mathbf{g}\right]. \tag{4.7}$$

It is also possible to perform a similar derivation for generalized Kerr-Schild vector fields if the signature admits null vectors. The result is

$$\varphi_s^*\mathbf{g} = \exp\left[\int_0^s dt\ \alpha(\varphi_t)\right]\left(\mathbf{g} + \int_0^s dt\ \exp\left[-\int_0^t dt^{'}\alpha(\varphi_{t'})\right] f^2(t)\beta(\varphi_t)\boldsymbol{k}\otimes\boldsymbol{k}\right).$$

$$\varphi_s^*\boldsymbol{k} = \boldsymbol{k}\exp\left[\int_0^s dt\ \gamma(\varphi_t)\right], \quad f(t) = \exp\left[\int_0^t du\ \gamma(\varphi_u)\right].$$

We thus see that bi-conformal vector fields can be characterized as *generalized* conformal motions‡ of both projectors P_{ab} and Π_{ab} of (2.3) which states clearly the geometric interpretation of these vector fields. This interpretation will be further supported when we study the highest dimension of finite-dimensional Lie algebras of bi-conformal vector fields in section 6 (see below for the definition of Lie algebras of bi-conformal vector fields). This characterization and equations (4.4), (4.5) lead us to the definition of *bi-conformal transformation*

‡ They are "generalized" in the sense that the conformal factors and gauges are functions on V, that is to say, they depend on *all* coordinates.

Definition 4.1 *A diffeomorphism $\Phi : V \to V$ is called a bi-conformal transformation if there exists a pair of orthogonal projectors P_{ab} and Π_{ab} with $\mathrm{g}_{ab} = P_{ab} + \Pi_{ab}$ such that*

$$\Phi^* P_{ab} = \lambda_1 P_{ab}, \quad \Phi^* \Pi_{ab} = \lambda_2 \Pi_{ab},$$

for some functions $\lambda_1,\ \lambda_2 \in C^1(V)$. Equivalently, this can be rewritten in terms of the square root $S_{ab} = P_{ab} - \Pi_{ab}$ as

$$\Phi^* \mathrm{g}_{ab} = \rho_1 \mathrm{g}_{ab} + \rho_2 S_{ab}, \quad \Phi^* S_{ab} = \rho_2 \mathrm{g}_{ab} + \rho_1 S_{ab},$$

where $\rho_1 = (\lambda_1 + \lambda_2)/2$, $\rho_2 = (\lambda_1 - \lambda_2)/2$.

A clue for the interpretation of bi-conformal vector fields can be obtained by examining their effect on $\mathbf{g}$ in local coordinates adapted to $\vec{\boldsymbol{\xi}}$.

Proposition 4.1 *Let $\vec{\boldsymbol{\xi}}$ be a bi-conformal vector field and $\{x^1, x^i\}$, $i = 2, \ldots, n$ a local coordinate system adapted to $\vec{\boldsymbol{\xi}}$ (i.e. $\vec{\boldsymbol{\xi}} = \partial/\partial x^1$) around any non-fixed point of $\vec{\boldsymbol{\xi}}$. In these coordinates the metric takes the form*

$$\mathrm{g}_{ab} = e^A G^0_{ab}(x^i) + e^B G^1_{ab}(x^i), \tag{4.8}$$

where (i) G^0_{ab} and G^1_{ab} do not depend on x^1 (they are invariant by $\vec{\boldsymbol{\xi}}$), rank$(G^0_{ab}) = P^a_{\ a}$, rank$(G^1_{ab}) = n - P^a_{\ a}$ and $G^0_{a}{}^{c} G^1_{cb} = 0$.

(ii) A and B are functions satisfying $\partial_{x^1} A = \alpha + \beta$, $\partial_{x^1} B = \alpha - \beta$. Furthermore the square root $\mathbf{S}$ is given in these coordinates by

$$S_{ab} = e^A G^0_{ab}(x^i) - e^B G^1_{ab}(x^i). \tag{4.9}$$

Proof : As is known, local coordinates such that $\vec{\boldsymbol{\xi}} = \partial/\partial x^1$ can always be chosen in a neighbourhood of any point in which $\vec{\boldsymbol{\xi}}$ does not vanish. In this local coordinate system equations (4.1) become

$$\frac{\partial}{\partial x^1}(\mathbf{g} + \mathbf{S}) = (\alpha + \beta)(\mathbf{g} + \mathbf{S}), \quad \frac{\partial}{\partial x^1}(\mathbf{g} - \mathbf{S}) = (\alpha - \beta)(\mathbf{g} - \mathbf{S}), \tag{4.10}$$

which can be explicitly integrated giving in components

$$\mathrm{g}_{ab}(x^1, x^i) + S_{ab}(x^1, x^i) = 2G^0_{ab}(x^i) \exp\left[\int_0^{x^1} ds(\alpha(s, x^i) + \beta(s, x^i))\right],$$

$$\mathrm{g}_{ab}(x^1, x^i) - S_{ab}(x^1, x^i) = 2G^1_{ab}(x^i) \exp\left[\int_0^{x^1} ds(\alpha(s, x^i) - \beta(s, x^i))\right].$$

Addition and subtraction of these equations leads to (4.8)-(4.9) with the obvious definitions for A and B. G^0_{ab} and G^1_{ab} are proportional to P_{ab} and Π_{ab} respectively from what we deduce the stated properties about their rank and product at once. $\square$

We finish this section pointing out that it is possible to define *conserved quantities and currents* for bi-conformal vector fields in a similar way as it has been done with other symmetry transformations (see [16, 24] for further details about this).

5. Lie algebras of bi-conformal vector fields

In this section we will settle under what circumstances bi-conformal vector fields give rise to a subalgebra of $\mathfrak{X}(V)$ and derive some of its basic properties. For a fixed square root $\mathbf{S}$ of the metric $\mathbf{g}$ we denote by $\mathcal{G}(\mathbf{S})$ the set of bi-conformal vector fields whose differential condition (3.1) involves $\mathbf{S}$. Bi-conformal vector fields belonging to $\mathcal{G}(\mathbf{S}_1)$ and $\mathcal{G}(\mathbf{S}_2)$ for two different square roots $\mathbf{S}_1$ and $\mathbf{S}_2$ are necessarily conformal motions of the metric $\mathbf{g}$ as we are going to prove now. A lemma is needed first.

Lemma 5.1 *If $\mathbf{S}_1$ and $\mathbf{S}_2$ are square roots of the metric $\mathbf{g}$ such that $\mathbf{S}_1 \times \mathbf{S}_2 = \lambda \mathbf{g}$ then $\mathbf{S}_1 = \pm \mathbf{S}_2$.*

Proof : From the assumptions we easily get

$$\mathbf{S}_1 \times \mathbf{S}_2 \times \mathbf{S}_2 = \lambda \mathbf{S}_2 \Rightarrow \mathbf{S}_1 = \lambda \mathbf{S}_2,$$

and using now $\mathbf{S}_1 \times \mathbf{S}_1 = \mathbf{S}_2 \times \mathbf{S}_2 = \mathbf{g}$ we conclude that $\lambda^2 = 1$. □

Proposition 5.1 *For two nonproportional square roots $\mathbf{S}_1$ and $\mathbf{S}_2$, $\mathcal{G}(\mathbf{S}_1) \cap \mathcal{G}(\mathbf{S}_2)$ is a set of conformal Killing vector fields.*

Proof : The existence of a non vanishing vector field $\vec{\boldsymbol{\xi}}$ belonging to $\mathcal{G}(\mathbf{S}_1) \cap \mathcal{G}(\mathbf{S}_2)$ entails the relation

$$\pounds_{\vec{\boldsymbol{\xi}}}\, \mathbf{g} = \alpha_1 \mathbf{g} + \beta_1 \mathbf{S}_1 = \alpha_2 \mathbf{g} + \beta_2 \mathbf{S}_2 \Rightarrow\ (\alpha_1 - \alpha_2)\mathbf{g} = \beta_2 \mathbf{S}_2 - \beta_1 \mathbf{S}_1, \tag{5.1}$$

Let us assume first that $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Performing the left inner product and the right inner product of the last equation with $\mathbf{S}_1$ we get respectively

$$\beta_1 \mathbf{g} - \beta_2 \mathbf{S}_1 \times \mathbf{S}_2 = (\alpha_2 - \alpha_1)\mathbf{S}_1,\ \beta_1 \mathbf{g} - \beta_2 \mathbf{S}_2 \times \mathbf{S}_1 = (\alpha_2 - \alpha_1)\mathbf{S}_1,$$

whose subtraction yields

$$\beta_2(\mathbf{S}_1 \times \mathbf{S}_2 - \mathbf{S}_2 \times \mathbf{S}_1) = 0 \Rightarrow \mathbf{S}_1 \times \mathbf{S}_2 = \mathbf{S}_2 \times \mathbf{S}_1$$

On the other hand squaring each member of (5.1) and using $\mathbf{S}_2 \times \mathbf{S}_2 = \mathbf{g}$ we obtain

$$-2\beta_1\beta_2 \mathbf{S}_1 \times \mathbf{S}_2 = ((\alpha_1 - \alpha_2)^2 - \beta_1^2 - \beta_2^2)\mathbf{g}.$$

Hence, $\mathbf{S}_1 \times \mathbf{S}_2$ is proportional to the metric tensor which is only possible if $\mathbf{S}_1 = \pm \mathbf{S}_2$ as we proved in lemma 5.1 contradicting the statement of the proposition. Thus some of the functions β_1 or β_2 must vanish which implies according to equation (5.1) that either $\mathbf{S}_1 = \pm \mathbf{g}$ and $\mathbf{S}_2 = \pm \mathbf{g}$, or $\beta_1 = \beta_2 = 0$, $\vec{\boldsymbol{\xi}}$ being in all of these possibilities a conformal Killing vector. □

The main conclusion of the previous proposition is the non-existence of a nontrivial proper bi-conformal vector field constructed from two different square roots $\mathbf{S}_1$ and $\mathbf{S}_2$, or in other words $\vec{\boldsymbol{\xi}}$ cannot leave invariant two different pairs of complementary projectors $\{P^a{}_b, \Pi^a{}_b\}$ and $\{P'^a{}_b, \Pi'^a{}_b\}$ (unless it is a conformal motion).

Our next result proves that $\mathcal{G}(\mathbf{S})$ is a Lie algebra for any square root $\mathbf{S}$.

Proposition 5.2 *The set of vector fields in $\mathcal{G}(\mathbf{S})$ with $\mathbf{S}$ a square root of the metric is a Lie subalgebra of the Lie algebra $\mathfrak{X}(V)$.*

Proof : We must show that the linear combination and Lie bracket of any pair of vectors $\vec{\boldsymbol{\xi}}_1$ and $\vec{\boldsymbol{\xi}}_2$ satisfying equations (3.1) for a fixed square root $\mathbf{S}$ is also a solution of this same pair of equations for different gauges. Clearly the gauge functions must depend on the chosen bi-conformal vector field $\vec{\boldsymbol{\xi}}$ in a precise fashion and we will write this dependence as $\alpha_{\vec{\boldsymbol{\xi}}}$, $\beta_{\vec{\boldsymbol{\xi}}}$. After a simple calculation, we get that $\vec{\boldsymbol{\xi}}_1+\vec{\boldsymbol{\xi}}_2$ and $[\vec{\boldsymbol{\xi}}_1,\vec{\boldsymbol{\xi}}_2]$ are bi-conformal vector fields with gauge functions given by

$$\alpha_{[\vec{\boldsymbol{\xi}}_1,\vec{\boldsymbol{\xi}}_2]}=\pounds_{\vec{\boldsymbol{\xi}}_1}\alpha_{\vec{\boldsymbol{\xi}}_2}-\pounds_{\vec{\boldsymbol{\xi}}_2}\alpha_{\vec{\boldsymbol{\xi}}_1},\quad \beta_{[\vec{\boldsymbol{\xi}}_1,\vec{\boldsymbol{\xi}}_2]}=\pounds_{\vec{\boldsymbol{\xi}}_1}\beta_{\vec{\boldsymbol{\xi}}_2}-\pounds_{\vec{\boldsymbol{\xi}}_2}\beta_{\vec{\boldsymbol{\xi}}_1},$$
$$\alpha_{\vec{\boldsymbol{\xi}}_1+\vec{\boldsymbol{\xi}}_2}=\alpha_{\vec{\boldsymbol{\xi}}_1}+\alpha_{\vec{\boldsymbol{\xi}}_2},\quad \beta_{\vec{\boldsymbol{\xi}}_1+\vec{\boldsymbol{\xi}}_2}=\beta_{\vec{\boldsymbol{\xi}}_1}+\beta_{\vec{\boldsymbol{\xi}}_2}.$$

□

Corollary 5.1 *The subspace $\mathcal{G}(\mathbf{S}_1)\cap\mathcal{G}(\mathbf{S}_2)$ is a Lie subalgebra of conformal Killing vectors for every pair of square roots $\mathbf{S}_1$ and $\mathbf{S}_2$.*

Proof : This is a consequence of proposition 5.2 and the fact that the intersection of two Lie algebras is a Lie subalgebra. □

We analyze next some properties of the Lie algebra $\mathcal{G}(\mathbf{S})$ which might shed some light on the nature of bi-conformal vector fields. First of all it is clear that $\mathcal{G}(\mathbf{S})$ may sometimes consist only on the trivial solution and thus $\mathcal{G}(\mathbf{S})=\{\vec{\mathbf{0}}\}$ which is a case of no interest. When $\mathcal{G}(\mathbf{S})\neq\{\vec{\mathbf{0}}\}$ an important question concerns their dimensionality as vector spaces. As happens with other known transformations in Differential Geometry, this dimension can be either finite or infinite depending on whether one is able to get a normal system of partial differential equations out of (3.1). Here normal means that the first derivatives of a well-defined set of unknowns can be isolated in terms of themselves, see [20] for a discussion and proof of this result and section 6. Nonetheless the search of a normal system to discard the existence of infinite dimensional Lie algebras can sometimes be bypassed if we state under what circumstances $\vec{\boldsymbol{\xi}}$ and $\rho\vec{\boldsymbol{\xi}}$ are bi-conformal vector fields for a function ρ. We settle this issue in the following proposition.

Proposition 5.3 *If both vector fields $\vec{\boldsymbol{\xi}}$ and $\rho\vec{\boldsymbol{\xi}}$ are in $\mathcal{G}(\mathbf{S})$ for some non-constant function ρ then $\boldsymbol{\Omega}$ is either a 1-form with $\boldsymbol{\Omega}\propto\boldsymbol{\xi}$ or a 2-form $\boldsymbol{\Omega}\propto\boldsymbol{\xi}\wedge d\rho$ where $\boldsymbol{\Omega}$ is the simple form generating the square root $\mathbf{S}$.*

Proof : To prove this we will study when the vector field $\rho\vec{\boldsymbol{\xi}}$ solves (3.1) given that $\vec{\boldsymbol{\xi}}$ does, being ρ a smooth function. Substitution of this in (3.1) yields ($\rho_a=\partial_a\rho$)

$$\rho_a\xi_b+\rho_b\xi_a=(\bar{\alpha}-\rho\alpha)\mathrm{g}_{ab}+(\bar{\beta}-\rho\beta)S_{ab} \tag{5.2}$$
$$\rho_a\xi_cS^c{}_b+\rho_b\xi_cS^c{}_a=(\bar{\beta}-\rho\beta)\mathrm{g}_{ab}+(\bar{\alpha}-\rho\alpha)S_{ab}, \tag{5.3}$$

where $\bar{\alpha}$, $\bar{\beta}$ are the gauges of $\rho\vec{\boldsymbol{\xi}}$. Multiplying by $S^b{}_c$ we get after a suitable relabelling of indexes

$$\rho_a\xi_cS^c{}_b+\xi_a\rho_cS^c{}_b=(\bar{\alpha}-\rho\alpha)S_{ab}+(\bar{\beta}-\rho\beta)\mathrm{g}_{ab}, \tag{5.4}$$
$$\rho_a\xi_b+\xi_eS^e{}_a\rho_cS^c{}_b=(\bar{\beta}-\rho\beta)S_{ab}+(\bar{\alpha}-\rho\alpha)\mathrm{g}_{ab}. \tag{5.5}$$

Subtracting (5.4) from (5.3), and (5.5) from (5.2) we get

$$\xi_a\rho_cS^c{}_b-\rho_b\xi_cS^c{}_a=0,\qquad \rho_cS^c{}_b\xi_eS^e{}_a-\rho_b\xi_a=0, \tag{5.6}$$

which can be reduced to a couple of linear homogeneous systems by contracting with ρ_cS^{cb} and ξ^a both relations

$$\left.\begin{array}{r}\xi_a(\rho_c\rho^c)-(\rho_cS^{cb}\rho_b)S^e{}_a\xi_e=0\\ \rho_c\rho^c\xi_eS^e{}_a-\rho^cS_{cb}\rho^b\xi_a=0\end{array}\right\},\qquad \left.\begin{array}{r}(\xi^a\xi_a)\rho_cS^c{}_b-\rho_b(\xi_cS^c{}_a\xi^a)=0\\ (\xi^eS_{ec}\xi^c)\rho_qS^q{}_b-(\xi_c\xi^c)\rho_b=0\end{array}\right\}. \tag{5.7}$$

The fulfillment of both systems implies that $\xi_b S^b{}_c = \epsilon\xi_c$ and $\rho_b S^b{}_c = \epsilon\rho_c$ with $\epsilon^2 = 1$. These conditions entail a further relation arising from (5.3) and (5.5) between the barred and unbarred gauges given by $\bar{\alpha} - \rho\alpha = \epsilon(\bar{\beta} - \rho\beta)$. From this we deduce that (5.2) takes the form

$$\rho_a\xi_b + \rho_b\xi_a = (\bar{\alpha} - \rho\alpha)(\mathrm{g}_{ab} + \epsilon S_{ab}).$$

Since the tensors $(\mathrm{g}_{ab} + \epsilon S_{ab})/2$ are projectors (either P_{ab} or Π_{ab}) they can be thought of as the metric tensor of a certain subspace (the subspace generated by the vectors forming the simple form generating $\mathbf{S}$) meaning that $d\rho \otimes \boldsymbol{\xi} + \boldsymbol{\xi} \otimes d\rho$ can only be algebraically such a projector if either the subspace is one-dimensional generated by $\boldsymbol{\xi}$ with $\boldsymbol{\xi} \propto d\rho$ or two dimensional with $d\rho$ and $\boldsymbol{\xi}$ as generators. If $\epsilon = 1$ this projector is P_{ab} so S_{ab} is the superenergy tensor of a simple form proportional to $\boldsymbol{\xi}$ in the one-dimensional case or to $\boldsymbol{\xi} \wedge d\rho$ in the two dimensional case (see considerations coming after equation (2.3)). The discussion for the case $\epsilon = -1$ is similar replacing P_{ab} by Π_{ab}. □

Remark. This result can be extended to include the dual of $\boldsymbol{\Omega}$, which is either a $(n-1)$-form or a $(n-2)$-form, by means of the property

$$\mathbf{T}\{*\boldsymbol{\Omega}\} = (-1)^{n-1}\frac{|\det(\mathbf{g})|}{\det(\mathbf{g})}\mathbf{T}\{\boldsymbol{\Omega}\}$$

holding for any superenergy tensor defined by equation (2.1).

We see as an outcome of this proposition that the cases with $P^a{}_a = 1, 2, n-1, n-2$ may contain infinite-dimensional Lie algebras $\mathcal{G}(\mathbf{S})$. This is what one should expect given the conformal character of these symmetries in the subspaces on which P_{ab} and Π_{ab} project because either of these subspaces will be of dimensions less or equal than two whenever $\mathbf{S}$ is built up from a 1-form or a 2-form (or their duals) and the conformal group in these dimensions can be of infinite dimension as is well known [43].

Another interesting result coming up from the proof of this proposition is that, in the case of $\mathbf{S} = T\{\boldsymbol{\Omega}\}$ for a 1-form $\boldsymbol{\Omega}$, this 1-form must be proportional to $\boldsymbol{\xi} \propto d\rho$ from what we conclude that $\vec{\boldsymbol{\xi}}$ is irrotational. Indeed the converse of this statement is also true.

Proposition 5.4 *Let $\vec{\boldsymbol{\xi}}$ be a bi-conformal vector field such that $\mathbf{S} \propto \mathbf{T}\{\boldsymbol{\xi}\}$. Then $\rho\vec{\boldsymbol{\xi}}$ is also a bi-conformal vector field iff $d\rho \wedge \boldsymbol{\xi} = 0$.*

Proof : The "if" is a particular case of proposition 5.3 so we are only left with the "only if" implication. Under the hypotheses of this proposition the tensor S_{ab} takes the form (equation (2.1) with $\boldsymbol{\Omega} = \boldsymbol{\xi}$)

$$S_{ab} = -\mathrm{g}_{ab} + 2\xi^{-2}\xi_a\xi_b, \quad \xi^2 \equiv \xi_a\xi^a.$$

If (3.1) is assumed, the first equation of the couple becomes

$$\nabla_a\xi_b + \nabla_b\xi_a = (\alpha - \beta)\mathrm{g}_{ab} + 2\xi^{-2}\beta\xi_a\xi_b. \tag{5.8}$$

Furthermore the second equation of (3.1) is a consequence of this because

$$\pounds_{\vec{\boldsymbol{\xi}}}\,\xi_a = \pounds_{\vec{\boldsymbol{\xi}}}\,(\mathrm{g}_{ap}\xi^p) = (\alpha\mathrm{g}_{ab} + \beta S_{ab})\xi^b = (\alpha + \beta)\xi_a \implies$$
$$\implies \pounds_{\vec{\boldsymbol{\xi}}}\,S_{ab} = -\pounds_{\vec{\boldsymbol{\xi}}}\,\mathrm{g}_{ab} + 2\pounds_{\vec{\boldsymbol{\xi}}}\,\left(\xi^{-2}\xi_a\xi_b\right) =$$
$$= -\alpha\mathrm{g}_{ab} - \beta S_{ab} - 2\xi^{-2}(\alpha + \beta)\xi_a\xi_b + 4\xi^{-2}(\alpha + \beta)\xi_a\xi_b = \alpha\left(-\mathrm{g}_{ab} + 2\xi^{-2}\xi_a\xi_b\right) + \beta\mathrm{g}_{ab},$$

so we only need to care about (5.8). The 1-form $\rho\xi_a$ fulfills the equation

$$\nabla_a(\rho\xi_b) + \nabla_b(\rho\xi_a) = (\bar{\alpha} - \bar{\beta})\mathrm{g}_{ab} + 2\bar{\beta}(\rho\xi)^{-2}\rho\xi_a\rho\xi_b = (\bar{\alpha} - \bar{\beta})\mathrm{g}_{ab} + 2\xi^{-2}\bar{\beta}\xi_a\xi_b$$

as long as $\gamma\xi_b = \partial_b\rho$ for some smooth function γ. The gauges $\bar{\alpha}$ and $\bar{\beta}$ are given then by $\bar{\beta} = \xi^2\gamma + \beta\rho$ and $\bar{\alpha} = \xi^2\gamma + \alpha\rho$. □

6. Normal system and maximal Lie algebras of bi-conformal vector fields

We start now to tackle the integrability conditions of equations (3.1) for a fixed Lorentz tensor as well as the greatest dimension of the vector space $\mathcal{G}(\mathbf{S})$ when it happens to be finite dimensional. First of all, we give an overview of the procedure to be followed which is quite similar to the one used to find out the first integrability conditions and the largest dimension of Lie algebras for isometries, conformal motions or collineations (see e.g. [43] for an account of this). The aim is to rewrite the equations satisfied by $\vec{\boldsymbol{\xi}}$, by using successive derivatives and identities, in *normal* form, that is to say, such that one can identify a definite set of unknowns, say Z_A, for which the equations become a set of first order PDEs like (6.3), see below, with all the derivatives isolated. In our case, this normal system, will actually be linear and homogeneous, in case it exists, as we are going to prove.

One starts with the differential conditions fulfilled by each generator of the symmetry under study written generically in the form,

$$\Phi_I(\vec{\boldsymbol{\xi}}, \nabla\vec{\boldsymbol{\xi}}, \ldots, \phi_1, \nabla\phi_1, \ldots) = 0, \tag{6.1}$$

where $\phi_1, \ldots$ are some scalar functions accounting for the gauges of the symmetry and the index I denotes the whole set of tensor indexes which appear in the differential conditions. These equations must be differentiated a number of times resulting in new algebraic equations involving the variables appearing in (6.1) plus higher derivatives of them (we now use the subindex I_k to gather the resulting new indexes)

$$\Phi^{(k)}_{I_k}(\vec{\boldsymbol{\xi}}, \nabla\vec{\boldsymbol{\xi}}, \nabla^2\vec{\boldsymbol{\xi}}, \ldots, \phi_1, \nabla\phi_1, \nabla^2\phi_1, \ldots) = 0, \; k = 1, 2, \ldots. \tag{6.2}$$

From these equations one wants to get another set which contains the first derivatives of the *system variables* isolated in terms of themselves and the manifold data. To do this one may need to include higher derivatives of the initial variables $\vec{\boldsymbol{\xi}}$, $\phi_1, \ldots$ as new independent variables so these definitions will be part of the normal system. Hence the normal set of equations will in general look like

$$D_a Z_A = f_{aA}(x, Z), \tag{6.3}$$

where $x = (x^1, \ldots, x^n)$ is the chosen local coordinate system, $Z = \{Z_A\} = (Z_1(x), \ldots, Z_m(x))$ denotes the complete set of system variables, D_a is a differential operator and f_{aA} are functions depending on the coordinates and the system variables. A normal system of equations can only be achieved for finite-dimensional Lie groups. An important point regarding this calculation is that the chain of equations (6.2) used to get the normal system may give rise to constraints between the system variables Z_A in the form

$$L_C(x, Z) = 0, \quad C = 1, \ldots, q. \tag{6.4}$$

These constraints arise when some of the equations of the above chain (or some linear combination of them) do not contain derivatives of the system variables.

The first integrability conditions are new relations between the system variables coming from the compatibility of the anti-symmetrized second derivatives $D_{[a}D_{b]}$ and the normal system (6.3)

$$2D_{[b}D_{a]}Z_A = D_b f_{aA}(x,Z) + \sum_C \frac{\partial f_{aA}(x,Z)}{\partial Z_C} f_{bC}(x,Z) - \\ -D_a f_{bA}(x,Z) - \sum_C \frac{\partial f_{bA}(x,Z)}{\partial Z_C} f_{aC}(x,Z), \tag{6.5}$$

which can be arranged in the same form as (6.4) where some identity for $D_{[a}D_{b]}$ must be used in this last step (for instance if D_a is the covariant derivative, the Ricci identity (6.13)). The constraints (6.4) themselves must be differentiated (propagated)

$$D_a L_C(x,Z) + \sum_B \frac{\partial L_C(x,Z)}{\partial Z_B} D_b Z_B = 0, \tag{6.6}$$

getting new relations involving the system variables and the data which are the first integrability conditions coming from the constraints. They must be added to the set arising from the commutation of the derivatives. The whole set of first integrability conditions will be identically satisfied if the $(V, \mathbf{g})$ we deal with is maximal with respect to the symmetry under study. The fulfillment of the maximal integrability will pose certain geometric conditions which characterize these maximal spaces. In maximal spaces arising from a normal form, the largest dimension of the Lie algebra of vector fields satisfying (6.1) is achieved, and this largest dimension is given by the total number of system variables m minus the number of independent constraints q found in (6.4). The set of solutions of (6.1) is a subspace of $\mathfrak{X}(V)$ which thus depends linearly on $m-q$ arbitrary constants. If the first integrability conditions are not identically satisfied, we must carry on the above described procedure but now applied to the first integrability conditions obtaining a chain of equations of the same type

$$\Xi^j(x,Z) = 0, \quad j = q+1, \dots \tag{6.7}$$

This new set of integrability conditions impose further geometric constraints for each j (we are assuming that all equations in (6.7) are algebraically independent). If there exists a value of j such that the corresponding condition is identically satisfied (or we are able to settle the geometric conditions for this to happen) then the solution of the differential conditions is a Lie subalgebra of vector fields of dimension $m-j$. On the other hand, if the number of linearly independent equations in (6.7) and (6.4) is greater or equal than m then we get a homogeneous linear system for the system variables with no solution other than the trivial one.

6.1. Normal system

In what follows, we are going to construct the normal system form for bi-conformal vector fields —in the cases this is possible— thereby obtaining also the largest dimension of some finite dimensional $\mathcal{G}(\mathbf{S})$ by following the above outlined procedure. This will in particular allow us to prove rigorously that the cases identified in the previous section are the only ones with a feasible infinite dimension. This is a rather long calculation and only its main excerpts are shown. To start with, we rewrite (3.1) (or (4.1)) in terms of the projectors P_{ab} and Π_{ab} as

$$\pounds_{\vec{\xi}} P_{ab} = \phi P_{ab}, \quad \pounds_{\vec{\xi}} \Pi_{ab} = \chi \Pi_{ab}, \tag{6.8}$$

$$\Rightarrow \quad \pounds_{\vec{\xi}} P^a{}_b = 0, \quad \pounds_{\vec{\xi}} \Pi^a{}_b = 0,$$
$$\Rightarrow \quad \pounds_{\vec{\xi}} P^{ab} = -\phi P^{ab}, \pounds_{\vec{\xi}} \Pi^{ab} = -\chi \Pi^{ab}$$

where $\phi = (\alpha + \beta)$ and $\chi = (\alpha - \beta)$ and the second pair is (4.2). The following well known formulae in differential geometry are needed (see [43] for an account of them)

$$\pounds_{\vec{\xi}} \gamma^a_{bc} = \frac{1}{2} g^{ae} \left[\nabla_b (\pounds_{\vec{\xi}} g_{ce}) + \nabla_c (\pounds_{\vec{\xi}} g_{be}) - \nabla_e (\pounds_{\vec{\xi}} g_{bc}) \right], \tag{6.9}$$

$$\pounds_{\vec{\xi}} \gamma^a_{bc} = \nabla_b \nabla_c \xi^a + \xi^d R^a{}_{cdb}, \tag{6.10}$$

$$\nabla_c \pounds_{\vec{\xi}} T^{a_1 \dots a_p}{}_{b_1 \dots b_q} - \pounds_{\vec{\xi}} \nabla_c T^{a_1 \dots a_p}{}_{b_1 \dots b_q} =$$
$$= -\sum_{j=1}^{p} (\pounds_{\vec{\xi}} \gamma^{a_j}_{cr}) T^{\dots a_{j-1} r a_{j+1} \dots}{}_{b_1 \dots b_q} + \sum_{j=1}^{q} (\pounds_{\vec{\xi}} \gamma^r_{cb_j}) T^{a_1 \dots a_p}{}_{\dots b_{j-1} r b_{j+1} \dots}, \tag{6.11}$$

$$\pounds_{\vec{\xi}} R^d{}_{cab} = \nabla_a (\pounds_{\vec{\xi}} \gamma^d{}_{bc}) - \nabla_b (\pounds_{\vec{\xi}} \gamma^d_{ac}), \tag{6.12}$$

where γ^a_{bc} is the Levi-Civita connection of the metric g_{ab} and $R^d{}_{cab}$ its curvature tensor. Our convention for the Riemann tensor is such that the Ricci identity is

$$\nabla_a \nabla_b u_c - \nabla_b \nabla_a u_c = u_d R^d{}_{cba}. \tag{6.13}$$

The Lie derivative of the connection can be worked out at once since we know $\pounds_{\vec{\xi}} \mathbf{g}$. In terms of the quantities defined in (6.8) it becomes ($\phi_b \equiv \partial_b \phi,\ \chi_b \equiv \partial_b \chi$)

$$\pounds_{\vec{\xi}} \gamma^a_{bc} = \frac{1}{2} (\phi_b P^a{}_c + \phi_c P^a{}_b - \phi^a P_{bc} + \chi_b \Pi^a{}_c + \chi_c \Pi^a{}_b - \chi^a \Pi_{cb} + (\phi - \chi) M^a{}_{bc}), \tag{6.14}$$

where we have defined the tensor

$$M^a{}_{bc} = \nabla_b P^a{}_c + \nabla_c P^a{}_b - \nabla^a P_{cb} = -(\nabla_b \Pi^a{}_c + \nabla_c \Pi^a{}_b - \nabla^a \Pi_{bc}), \tag{6.15}$$
$$M^a{}_{bc} = M^a_{(bc)}, \ M^a{}_{ac} = 0,$$

coming the second equality in (6.15) from $\nabla_c P_{ab} = -\nabla_c \Pi_{ab}$. $M^a{}_{bc}$ can be equally defined from the square root $\mathbf{S}$ as

$$M^a{}_{bc} = \frac{1}{2} \left(\nabla_b S^a{}_c + \nabla_c S^a{}_b - \nabla^a S_{cb} \right).$$

Observe also that

$$P^{ca} M_{acb} = P^{ca} (\nabla_c P_{ab} + \nabla_b P_{ac} - \nabla_a P_{bc}) = P^{ca} \nabla_b P_{ac} = 0$$

as follows from $0 = \nabla_b (P^{ca} P_{ac})$, and analogously

$$\Pi^{ab} M_{abc} = 0.$$

Using (6.11) and (6.14) we can calculate the Lie derivative of M_{abc}

$$\pounds_{\vec{\xi}} M_{abc} = \phi M_{abc} + (\chi - \phi) P_{ap} M^p{}_{bc} - P_{bc} \Pi_{ap} \phi^p + \Pi_{cb} P_{ap} \chi^p$$
$$= \chi M_{abc} + (\phi - \chi) \Pi_{ap} M^p{}_{bc} - P_{bc} \Pi_{ap} \phi^p + \Pi_{cb} P_{ap} \chi^p, \tag{6.16}$$

where the identity $\phi M_{abc} + (\chi - \phi) P_{ap} M^p{}_{bc} = \chi M_{abc} + (\phi - \chi) \Pi_{ap} M^p{}_{bc}$ must be used to get the last equality. The trace of (6.16) can be split in two if we contract with P^{cb} and Π^{cb} respectively getting

$$\pounds_{\vec{\xi}} (M_{abc} P^{bc}) + (\phi - \chi) P_{ap} M^p{}_{bc} P^{bc} = -p \Pi_{ap} \phi^p$$
$$\pounds_{\vec{\xi}} (M_{abc} \Pi^{bc}) + (\chi - \phi) \Pi_{ap} M^p{}_{bc} \Pi^{bc} = -(n - p) P_{ap} \chi^p,$$

(recall that $p = P^a{}_a$). A further simplification of this arises if we realize the property $0 = P_{ap}M^p{}_{cb}P^{cb} = \Pi_{ap}M^p{}_{cb}\Pi^{cb}$ so the last couple of equations yields

$$\pounds_{\vec{\xi}}\, E_a = -p\Pi_{ap}\phi^p, \quad \pounds_{\vec{\xi}}\, W_a = (p-n)P_{ap}\chi^p, \tag{6.17}$$

where

$$E_a \equiv M_{acb}P^{cb}, \;\; W_a \equiv -M_{acb}\Pi^{cb}, \; \Pi_{ac}E^c = E_a, \; P_{ac}W^c = W_a, \; 0 = P^{ab}E_b = \Pi^{ab}W_b.$$

Now we can use equation (6.12) to work out $\pounds_{\vec{\xi}}\, R^d{}_{cab}$ obtaining

$$\pounds_{\vec{\xi}}\, R^d{}_{cab} = P^d_{[b}\nabla_{a]}\phi_c + P_{c[a}\nabla_{b]}\phi^d + \Pi^d{}_{[b}\nabla_{a]}\chi_c + \Pi_{c[a}\nabla_{b]}\chi^d + \phi_{[b}\nabla_{a]}P^d_c + \phi_c\nabla_{[a}P^d_{b]}$$
$$+\phi^d\nabla_{[b}P_{a]c} + \chi_{[b}\nabla_{a]}\Pi^d{}_c + \chi_c\nabla_{[a}\Pi^d{}_{b]} + \chi^d\nabla_{[b}\Pi_{a]c} + \nabla_{[a}[M^d{}_{b]c}(\phi-\chi)]. \tag{6.18}$$

Multiplying here by P^a_d, using algebraic properties and $\nabla_b\phi_c = \nabla_c\phi_b$ we get

$$2\pounds_{\vec{\xi}}\,(P^a_d R^d{}_{cab}) = -\nabla_c(\Pi^a{}_b\phi_a) - \nabla_b(\Pi^a{}_c\phi_a) - (\nabla_c P^a_b + \nabla_b P^a_c + P^{pa}\nabla_p P_{bc})\phi_a +$$
$$+(2-p)\,\nabla_b\phi_c - P^{ad}\nabla_a\phi_d P_{bc} - P^{ad}\nabla_a\chi_d\Pi_{cb} + \frac{1}{2}(\phi_b E_c + \phi_c E_b - \chi_b E_c - \chi_c E_b) -$$
$$-P^{ad}\chi_d\nabla_a\Pi_{cb} + P^a_d\{\nabla_a[(\phi-\chi)M^d{}_{bc}] - \nabla_b[(\phi-\chi)M^d{}_{ac}]\}. \tag{6.19}$$

A further contraction of this equation with P^{cb} yields

$$2(1-p)P^{cb}\nabla_b\phi_c = 2\pounds_{\vec{\xi}}\, R^0 + 2\phi R^0 + (\chi-\phi)W^0, \tag{6.20}$$

where $R^0 = P^{cb}P^{ar}R_{rcab}$ and $W^0 = P^{ar}(\nabla_a M_{rbc} - \nabla_b M_{rac})P^{cb}$. After substitution of (6.20) into (6.19) we realize that, in order to get an expression with the derivatives of ϕ_b and χ_b isolated, the only terms which require of a further treatment are

$$-\nabla_c(\phi_r\Pi^r{}_b) + \nabla_b(\phi_r\Pi^r{}_c) - P^{ad}\nabla_a\chi_d\Pi_{cb},$$

which can be worked out by derivating the second equation of (6.17) and using (6.11). The result of such calculation is

$$\nabla_c(\Pi_{ar}\phi^r) = -\frac{1}{p}[\pounds_{\vec{\xi}}\,(\nabla_c E_a) - \frac{1}{2}(E_r\phi^r)P_{ac} - \frac{1}{2}(E_r\chi^r)\Pi_{ac} + \chi_{(a}E_{c)} - \frac{1}{2}(\chi-\phi)E_r M^r{}_{ca}], \tag{6.21}$$

$$\nabla_c(P_{ar}\chi^r) = \frac{-1}{n-p}[\pounds_{\vec{\xi}}\,(\nabla_c W_a) - \frac{1}{2}(W_r\chi^r)\Pi_{ac} - \frac{1}{2}(W_r\phi^r)P_{ac} + \phi_{(c}W_{a)} - \frac{1}{2}(\chi-\phi)W_r M^r{}_{ca}]. \tag{6.22}$$

Equation (6.21) provides us the answer for the terms $-\nabla_c(\varphi_r\Pi^r{}_b)+\nabla_b(\varphi_r\Pi^r{}_c)$ whereas a further manipulation of (6.22) yields

$$P^{ad}\nabla_a\chi_d = \nabla_a(P^{ad}\chi_d) - \frac{1}{2}(E^d - W^d)\chi_d \quad \Longrightarrow$$
$$P^{ad}\nabla_a\chi_d = \frac{-1}{n-p}\pounds_{\vec{\xi}}\,(\nabla_a W^a) + W_r\chi^r + \frac{p-2}{2(n-p)}W_r\phi^r - \frac{1}{2}E^r\chi_r - \frac{1}{n-p}\phi\nabla_r W^r. \tag{6.23}$$

Therefore substitution of (6.23), (6.21) and (6.20) into (6.19) results, after some tedious algebra, in

$$\nabla_c\phi_b = \frac{1}{2-p}\pounds_{\vec{\xi}}\,\left[2P^{ad}R_{dcab} - \frac{2}{p}\nabla_{(c}E_{b)} + \frac{R^0}{1-p}P_{bc} - \frac{1}{n-p}\nabla_r W^r\Pi_{cb}\right] +$$
$$+\left(\frac{1}{2p}E_r\chi^r - \frac{1}{2(n-p)}W_r\phi^r + \frac{1}{2-p}W_r\chi^r\right)\Pi_{cb} - \frac{1}{p}\chi_{(c}E_{b)} - \frac{1}{2-p}\phi_{(b}E_{c)} +$$

$$+\left(\frac{2}{2-p}\nabla_{(c}P^r{}_{b)}+\frac{1}{p(2-p)}E^rP_{cb}\right)\phi_r+\frac{(\phi_r-\chi_r)}{2-p}(-2P^{dr}\nabla_{(b}P_{c)d}+2P^{pr}\nabla_pP_{bc})+$$
$$+\frac{(\chi-\phi)}{2-p}(P^{ad}\nabla_aM_{dbc}+\nabla_bP^{ad}\nabla_cP_{ad}+\frac{1}{p}E_rM^r{}_{cb}+\frac{W^0}{2(1-p)}P_{bc}+\frac{1}{n-p}\nabla_rW^r\Pi_{cb}). \qquad (6.24)$$

This equation has a counterpart obtained by means of the replacements $\phi\leftrightarrow\chi$, $P_{ab}\leftrightarrow\Pi_{ab}$ and $p\leftrightarrow n-p$ which we shall omit for the sake of brevity. Equation (6.24) and its counterpart together with the following ones

$$\begin{aligned}&\partial_a\phi=\phi_a\\&\partial_a\chi=\chi_a\\&\nabla_a\xi_b=\Psi_{ab}+\frac{1}{2}(\phi P_{ab}+\chi\Pi_{ab}),\quad \Psi_{[ab]}=\Psi_{ab}\\&\nabla_b\Psi_{ca}=\xi_dR^d{}_{bca}+\phi_{[c}P_{a]b}+\chi_{[c}\Pi_{a]b}+(\phi-\chi)\nabla_{[c}P_{a]b},\end{aligned} \qquad (6.25)$$

comprise the normal system, where the third equation is the first of (3.1) written in terms of the system variables and the formula for $\nabla_b\Psi_{ca}$ is (6.10) with the Lie derivatives of the connection replaced by the values given in (6.14).

6.2. Constraint equations

The variables appearing in equations (6.24), its counterpart, and (6.25) are not independent because, as we are going to show next, they must fulfill a certain set of constraints reducing the effective number of them. To realize the existence of such constraints, let us review the procedure we have followed to derive the normal system out of the original differential conditions. We started with the differential conditions (6.8), differentiated them obtaining (6.16) and the fourth equation of (6.25), and then we calculated the second covariant derivative of the differential conditions (which essentially is (6.18)) yielding (6.24) and its partner after some algebraic manipulations involving the differentiation of both equations in (6.17). Therefore the equations which play the role of the chain written in (6.2) are

- differential conditions . . . . (6.8).
- linear combination of first derivatives . . .(6.10), (6.17), $\phi_a=\partial_a\phi$, $\chi_a=\partial_a\chi$.
- linear combination of second derivatives . . . . . (6.19), (6.21), (6.22), (6.23),

because these and only these equations are used to get the normal system (6.24) and (6.25). Nonetheless, some of the previous equations (or suitable linear combinations) do not contain derivatives of the system variables when expanded in terms of them which means that they are constraints of type (6.4). These are

$$\xi^c\nabla_cS_{ab}+\Psi_{ac}S^c{}_b+\Psi_{bc}S^c{}_a=0,\quad S_{ab}=P_{ab}-\Pi_{ab} \qquad (6.26)$$

$$\xi^c\nabla_cE_a+\Psi_{ac}E^c=-\frac{1}{2}\chi E_a-\frac{1}{2}(n+p)\Pi_{ap}\phi^p, \qquad (6.27)$$

$$\xi^c\nabla_cW_a+\Psi_{ac}W^c=-\frac{1}{2}\phi W_a-\frac{1}{2}(n-p)P_{ap}\chi^p, \qquad (6.28)$$

where the first constraint is the second of (3.1) (equivalently a linear combination of (6.8)) and (6.27), (6.28) are the first and second of (6.17) respectively. These constraints must be appended to (6.24), its partner, and (6.25) and they are needed to settle the true number of independent variables. In appendix A we will prove that the first expression only entails $p(n-p)$ independent equations whereas the next

couple contain $n-p$ and p independent equations respectively. This last statement can be seen for (6.27) if we note that ${P_a}^b E_b = 0$ which means that we have at most as many equations as the dimension of the subspace orthogonal to ${P_a}^b$, that is, $n-p$. The same reasoning gives p independent equations for (6.28), whence both equations amount for $n-p+p=n$ independent equations as claimed.

6.3. Maximal spaces

The full normal system will always make sense unless either of $2-p$, $1-p$, $2-n+p$, $1-n+p$ vanishes ($n=\pm p$ must be discarded here since it is only possible if either P_{ab} or Π_{ab} vanishes). We will see in the next subsection that these cases correspond with the ones found in proposition 5.3 but before doing that let us give the sought upper bound for the dimension of $\mathcal{G}(\mathbf{S})$ when there is a normal system. This number is the total number of variables appearing in the system (6.24), its counterpart and (6.25) minus the constraints (6.26), (6.27), (6.28)

counting	n	$C_{n,2}$	2	$2n$
variables	ξ_a	Ψ_{ab}	ϕ,χ	ϕ_r,χ_r

counting	$p(n-p)$	$n-p$	p
constraints	(6.26)	(6.27)	(6.28)

Thus an upper bound N for the dimension of $\mathcal{G}(\mathbf{S})$ is $N = n + C_{n,2} + 2 + 2n - p(n-p) - n = \frac{1}{2}[n^2 + n(3-2p) + 4 + 2p^2]$. The natural number N can be rewritten as $(p+1)(p+2)/2 + (n-p+1)(n-p+2)/2$. We have thus proven

Theorem 6.1 *If $p, n-p \notin \{1,2\}$, every Lie algebra of bi-conformal vector fields $\mathcal{G}(\mathbf{S})$ has finite dimension with*

$$dim(\mathcal{G}(\mathbf{S})) \leq \frac{1}{2}(p+1)(p+2) + \frac{1}{2}(n-p+1)(n-p+2) \equiv N.$$

The right hand side of this inequality is the sum of the maximum number of conformal motions in p dimensions plus the same number for $n-p$ dimensions suggesting that the maximum dimension of $\mathcal{G}(\mathbf{S})$ is achieved when our space can be split in two conformally flat dimensional pieces. We can show that this is true and that the maximum dimension can be actually realized.

Proposition 6.1 *The previously defined number N is the maximum dimension of $\mathcal{G}(\mathbf{S})$ if $p,\ n-p \notin \{1,2\}$ being this dimension attained for any $(V,\mathbf{g})$ whose line element is in local coordinates $\{x^a\}$, $a=1,\dots,n$*

$$ds^2 = \phi_1^2(x^a)\eta^0_{\alpha\beta}dx^\alpha dx^\beta + \phi_2^2(x^a)\eta^1_{AB}dx^A dx^B, \tag{6.29}$$

where $x^\alpha = \{x^1,\dots,x^p\}$, $x^A = \{x^{p+1},\dots,x^n\}$ are sets of coordinates and $\boldsymbol{\eta}^0$, $\boldsymbol{\eta}^1$ flat metrics of the appropriate signatures depending only on the coordinates $\{x^\alpha\}$ and $\{x^A\}$ respectively.

Remark. This result is local, valid in any neighbourhood in which the local coordinates are defined. The maximum number of *global* bi-conformal vector fields can be obviously less than N.

Proof : From equation (6.8) is clear that every conformal vector field of $\boldsymbol{\eta}^0$ or $\boldsymbol{\eta}^1$ is a bi-conformal vector field of $(V,\mathbf{g})$ because the projectors $\boldsymbol{P}$ and $\boldsymbol{\Pi}$ are given in terms of the flat metrics $\boldsymbol{\eta}^0$ and $\boldsymbol{\eta}^1$ by

$$\boldsymbol{P} = \phi_1^2\boldsymbol{\eta}^0, \quad \boldsymbol{\Pi} = \phi_2^2\boldsymbol{\eta}^1,$$

whence

$$\pounds_{\vec{\boldsymbol{\xi}}_1} \boldsymbol{P} = \left(\phi_1^{-2}\pounds_{\vec{\boldsymbol{\xi}}_1}(\phi_1^2) + \sigma_1\right)\boldsymbol{P}, \quad \pounds_{\vec{\boldsymbol{\xi}}_1} \mathbf{\Pi} = \phi_2^{-2}\pounds_{\vec{\boldsymbol{\xi}}_1}(\phi_2^2)\,\mathbf{\Pi},$$

where $\vec{\boldsymbol{\xi}}_1$ is any conformal Killing vector of the metric $\boldsymbol{\eta}^0$, that is, $\pounds_{\vec{\boldsymbol{\xi}}_1}\boldsymbol{\eta}^0 = \sigma_1\boldsymbol{\eta}^0$, $\pounds_{\vec{\boldsymbol{\xi}}_1}\boldsymbol{\eta}^1 = 0$. Similarly we can prove the result for conformal Killing vectors of the metric $\boldsymbol{\eta}^1$. Thus the line element given by equation (6.29) is maximally symmetric for bi-conformal vector fields since there are $\frac{1}{2}(p+1)(p+2)$ linearly independent conformal Killing vectors for the metric $\boldsymbol{\eta}^0$ and $\frac{1}{2}(n-p+1)(n-p+2)$ for the metric $\boldsymbol{\eta}^1$ being mutually linearly independent. □

Remark. Proposition 6.1 has an obvious generalized validity for the cases where p and/or $n-p$ are 1 or 2. The statement then is that *every conformal Killing vector of either of the two pieces* $\boldsymbol{\eta}^0$ *or* $\boldsymbol{\eta}^1$ *is a bi-conformal vector field of the space* $(V, \mathbf{g})$.

6.4. Infinite dimensional Lie algebras of bi-conformal vector fields

As we have already mentioned, the normal system given by the set (6.24), its partner, and (6.25) cannot be defined for some values of the trace $p = P^a_{\ a}$ which is the dimension of the subspace onto which $P^a_{\ b}$ projects. Therefore these values of p are linked to the possibility of infinite-dimensional Lie algebras of bi-conformal vector fields. Those values were

$$p = 2, \quad p = 1, \quad p = n-2, \quad p = n-1,$$

in which case (6.24) and its partner are not defined and there is no normal system for the variables ξ_a, Ψ_{ab}, ϕ, χ, ϕ_r and χ_r. In principle we cannot assure that a normal system does not exist because it may well happen that further derivatives of these equations are required to get it. Nonetheless, we proved in proposition 5.3 that, if the trace of $P^a_{\ b}$ takes one of the above values, infinite-dimensional Lie algebras of bi-conformal vector fields could be constructed from what we conclude that it does not exist a normal system for these values of p. We arrive thus at the following result

Proposition 6.2 *The only possible cases in which* $\mathcal{G}(\mathbf{S})$ *may be infinite dimensional take place when* $P^a_{\ b}$ *projects on a subspace whose dimension is either* 1*,* 2*,* $n-1$ *or* $n-2$*.*

Note that this completes proposition 5.3: infinite dimensional cases can only arise for square roots generated by 1-forms, 2-forms, $(n-1)$-forms or $(n-2)$-forms.

Corollary 6.1 *If* $n < 6$ *then every group of bi-conformal transformations is liable to be infinite dimensional.*

7. First integrability conditions: a preliminary analysis

In this section we turn our attention to finding part of the first integrability conditions of the normal system formed by (6.24), its counterpart, and (6.25) together with the constraints (6.26)-(6.28) according to the procedure outlined at the beginning of section 6. The calculation of the full integrability conditions of all the equations is long and it has not been completed yet. Nevertheless, we have reached a sufficiently advanced stage so that relevant information can already be extracted. In this sense, we will establish a necessary geometric condition for a line-element to adopt the form

(6.29) in local coordinates. In doing so we will use the calculations performed in the previous section.

Let us start with the integrability conditions arising from the constraint equations. The first of such integrability conditions comes from the combination of equations (6.17) with (6.16) yielding

$$\pounds_{\vec{\xi}}\left(M_{acb}-\frac{1}{p}E_aP_{bc}+\frac{1}{n-p}W_a\Pi_{cb}\right)=\phi\left(\Pi_{ap}M^p{}_{cb}-\frac{E_aP_{bc}}{p}\right)+ \\ +\chi\left(P_{ap}M^p{}_{bc}+\frac{\Pi_{cb}W_a}{n-p}\right). \tag{7.1}$$

Observe that (6.17) is the trace of (6.16) so (7.1) is the traceless part of (6.16). This calculation is equivalent to differentiating equation (6.26) and using the normal system so we can regard (7.1) as the first integrability condition of the constraint (6.26). For a better handling of some forthcoming expressions, let us define the tensor

$$T_{abc}\equiv M_{abc}+\frac{1}{n-p}W_a\Pi_{bc}-\frac{1}{p}E_aP_{bc}. \tag{7.2}$$

Straightforward properties are

$$P^{ab}T_{abc}=0,\ \Pi^{ab}T_{abc}=0,\quad P^{bc}T_{abc}=0,\quad \Pi^{bc}T_{abc}=0,$$

so T_{abc} is traceless in every index contraction. From this tensor we can define two other as

$$A_{abc}\equiv P_a{}^dT_{dbc}=P_a{}^dM_{dcb}+\frac{1}{n-p}W_a\Pi_{cb}, \tag{7.3}$$

$$B_{abc}\equiv \Pi_a{}^dT_{dbc}=\Pi_a{}^dM_{dcb}-\frac{1}{p}E_aP_{cb}, \tag{7.4}$$

which allow us to rewrite (7.1) in a number of equivalent ways

$$\pounds_{\vec{\xi}}\,T_{abc}=(\phi\Pi_a{}^s+\chi P_a{}^s)T_{sbc}\Longleftrightarrow \pounds_{\vec{\xi}}\,A_{abc}=\chi A_{abc},\ \pounds_{\vec{\xi}}\,B_{abc}=\phi B_{abc}\Longleftrightarrow \\ \Longleftrightarrow \pounds_{\vec{\xi}}\,A^a{}_{bc}=(\chi-\phi)A^a{}_{bc},\ \pounds_{\vec{\xi}}\,B^a{}_{bc}=(\phi-\chi)B^a{}_{bc}, \tag{7.5}$$

from what we deduce the invariances

$$\pounds_{\vec{\xi}}\,(A^a{}_{bc}B^d{}_{ef})=0,\ \pounds_{\vec{\xi}}\,(A_{abc}\Pi^{de})=0,\ \pounds_{\vec{\xi}}\,(B_{abc}P^{de})=0. \tag{7.6}$$

An important property of T_{abc} is shown next.

Theorem 7.1 *A sufficient condition such that the first integrability condition (7.1) of (6.26) is identically satisfied is that the tensor T_{abc} given in (7.2) vanishes identically.*

Proof : This follows from the the first of (7.5) straightforwardly. □

We will see later that the tensor T_{abc} actually vanishes if the metric tensor takes the form (6.29).

The integrability conditions of equations (6.27) and (6.28) present a lengthy form which shall be omitted in ths paper. Next we address the integrability conditions of (6.24-6.25). A not very long calculation proves that the integrability conditions of the first three equations of (6.25) are identically satisfied. This is evident for the first two and is reasonable for the third because the last of (6.25) is, in essence, its derivative. Therefore we are only left with the last of (6.25), (6.24) and its counterpart, whose first integrability conditions are calculated by means of the Ricci identity (6.13) applied to $\nabla_{[a}\nabla_{b]}\Psi_{cd}$, $\nabla_{[a}\nabla_{b]}\phi_c$ and $\nabla_{[a}\nabla_{b]}\chi_c$, respectively, and using the system itself to get rid of the first derivatives of the system variables. This last calculation and the geometric conditions imposed by the whole set of first integrability conditions are under current research and they will be presented elsewhere.

7.1. The condition $T_{abc} = 0$

The vanishing of the tensor T_{abc} is a sufficient condition for the integrability constraint (7.1) to be fulfilled as we proved in theorem 7.1. To investigate further the geometric significance of this condition, let us compute this tensor for metrics given in local coordinates $\{x^a\}$ by

$$ds^2 = \mathrm{g}_{ab}dx^a dx^b = \mathrm{g}_{\alpha\beta}dx^\alpha dx^\beta + \mathrm{g}_{AB}dx^A dx^B, \ \ \alpha, \beta = 1, \dots p, \ \ A, B = p+1, \dots, n, \quad (7.7)$$

where $\mathrm{g}_{\alpha\beta}$ and g_{AB} are functions of all the coordinates $\{x^a\}$. The tensors P_{ab} and Π_{ab} with components

$$P_{ab} = \mathrm{g}_{\alpha\beta}\delta^\alpha{}_a\delta^\beta{}_b, \quad \Pi_{ab} = \mathrm{g}_{AB}\delta^A{}_a\delta^B{}_b, \tag{7.8}$$

are orthogonal projectors, whose nonvanishing components are $P_{\alpha\beta} = \mathrm{g}_{\alpha\beta}$ and $\Pi_{AB} = \mathrm{g}_{AB}$, playing the role of (2.3) in this case. Thus, they can be used to calculate T_{abc} according to its definition (7.2). The non-zero components of the Christoffel symbols are

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\mathrm{g}^{\alpha\rho}(\partial_\beta \mathrm{g}_{\gamma\rho} + \partial_\gamma \mathrm{g}_{\rho\beta} - \partial_\rho \mathrm{g}_{\beta\gamma}), \ \Gamma^\alpha{}_{\beta A} = \frac{1}{2}\mathrm{g}^{\alpha\rho}\partial_A \mathrm{g}_{\beta\rho}, \ \Gamma^\alpha{}_{BA} = -\frac{1}{2}\mathrm{g}^{\alpha\rho}\partial_\rho \mathrm{g}_{BA},$$

$$\Gamma^A{}_{B\alpha} = \frac{1}{2}\mathrm{g}^{AD}\partial_\alpha \mathrm{g}_{BD}, \ \Gamma^A{}_{\alpha\beta} = -\frac{1}{2}\mathrm{g}^{AD}\partial_D \mathrm{g}_{\beta\alpha}, \ \Gamma^A{}_{BC} = \frac{1}{2}\mathrm{g}^{AD}(\partial_B \mathrm{g}_{CD} + \partial_C \mathrm{g}_{DB} - \partial_D \mathrm{g}_{BC}),$$

from what we conclude that the nonvanishing components of M_{abc}, E_a, and W_a are, respectively

$$M_{\alpha AB} = \partial_\alpha \mathrm{g}_{AB}, \quad M_{A\alpha\beta} = -\partial_A \mathrm{g}_{\alpha\beta},$$
$$E_A = -\partial_A \log|\det(\mathrm{g}_{\alpha\beta})|, \quad W_\alpha = -\partial_\alpha \log|\det(\mathrm{g}_{AB})|.$$

The condition $T_{abc} = 0$ entails the couple of partial differential equations

$$\partial_\alpha \mathrm{g}_{AB} = -\frac{1}{n-p}\,\mathrm{g}_{AB}W_\alpha, \quad \partial_A \mathrm{g}_{\alpha\beta} = -\frac{1}{p}\,\mathrm{g}_{\alpha\beta}E_A,$$

whose solution is

$$\mathrm{g}_{\alpha\beta} = G_{\alpha\beta}(x^\delta)e^{\Lambda_1(x^a)}, \ \mathrm{g}_{AB} = G_{AB}(x^D)e^{\Lambda_2(x^a)}, \tag{7.9}$$

where $G_{\alpha\beta}$, G_{AB}, Λ_1, Λ_2 are arbitrary functions of their respective arguments with no restrictions other than $\det(G_{\alpha\beta}) \neq 0$, $\det(G_{AB}) \neq 0$. We arrive thus at the following important result.

Theorem 7.2 *A necessary condition for the existence of a coordinate system in which a metric g_{ab} decomposes according to equation (7.7) with $\mathrm{g}_{\alpha\beta}$ and g_{AB} given by (7.9) is that the tensor T_{abc} defined in (7.2) vanishes identically.* □

Remark. Observe once again that this theorem holds even in the cases with p and/or $n-p$ taking the values 1 or 2.

The vanishing of the tensor T_{abc} is thus part of a possible invariant characterization of "breakable spaces", in the sense that the metric tensor decomposes according to (7.7) and (7.9). These spaces have been called *double-twisted products* in [34], as the "warping" or "twisting" functions Λ_1 and Λ_2 depend on all coordinates of the manifold. Particular interesting cases of these are (i) warped-product spaces where Λ_1, Λ_2 depend only on the $\{x^\alpha\}$ coordinates, say, (see e.g. [8], or [1] for a study of warped product spaces in Lorentzian geometry); (ii) conformally reducible spaces in which $e^{\Lambda_1} = e^{\Lambda_2}$ (see [11] for dimension four and Lorentzian signature), (iii) twisted-product spaces where Λ_1 (say) depends only on the coordinates x^α —so that

$g_{\alpha\beta}$ is a "true" metric on the $\{x^\alpha\}$-space—, see [13, 34, 21]; and (iv) double-warped product spaces, e.g. [34, 35], where Λ_1 depends only on the $\{x^A\}$-coordinates and vice versa for Λ_2. A major problem when dealing with such breakable spaces is their invariant characterization by means of some (local) criterion. Attempts towards this direction have been made for instance in [10, 11, 35].

In the particular case of $G_{\alpha\beta}$ and G_{AB} being conformally flat metrics we recover the case studied in proposition 6.1 and the space is maximal, namely, it admits a maximum number of bi-conformal vector fields.

8. Examples

Example 1. Our first example of bi-conformal vector field was already presented in [24]. We briefly mention it here due to its interest in Lorentzian geometry (see [24] for the definitions of the concepts and the proof).

Proposition 8.1 *Every causal-preserving vector field with $n-1$ linearly independent canonical null directions for $n \geq 3$ is a bi-conformal vector field with $\mathbf{S}$ the Lorentz tensor built up from the canonical null directions.*

Example 2. We present next an example of an algebra of bi-conformal vector fields which generalizes the one we gave in [23] for warped product spacetimes. A bi-conformal vector field is fixed once we set a simple p-form $\mathbf{\Omega} = \boldsymbol{\theta}_1 \wedge \ldots \wedge \boldsymbol{\theta}_p$ defining a square root $\mathbf{S}$. A simple choice is a differential p-form such that the distribution spanned by $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_p$ is integrable which, according to Frobenius theorem, happens if and only if $d\boldsymbol{\theta}_\alpha = \sum_{\beta=1}^{p} \boldsymbol{\omega}_{\alpha\beta} \wedge \boldsymbol{\theta}_\beta$ for a certain set of 1-forms $\boldsymbol{\omega}_{\alpha\beta}$. This means that $d\mathbf{\Omega} = \boldsymbol{\omega} \wedge \mathbf{\Omega}$, $\boldsymbol{\omega} = \boldsymbol{\omega}_{11} - \boldsymbol{\omega}_{22} + \ldots + (-1)^{p-1}\boldsymbol{\omega}_{pp}$ and we can set up a coordinate system $\{x^1, \ldots, x^n\}$ in which $\mathbf{\Omega}$ takes the form $\mathbf{\Omega} = \rho(x) dx^1 \wedge \ldots \wedge dx^p$. The line element in this coordinate system can be decomposed as

$$ds^2 = \mathrm{g}_{\alpha\beta} dx^\alpha dx^\beta + 2\mathrm{g}_{\alpha A} dx^\alpha dx^A + \mathrm{g}_{AB} dx^A dx^B, \quad 1 \leq \alpha, \beta \leq p, \ p+1 \leq A, B \leq n,$$

where $\det(\mathrm{g}_{\alpha\beta}) \neq 0$ and the signatures of $\mathrm{g}_{\alpha\beta}$ and g_{AB} are left free. The metric tensor components depend on all coordinates. Formula (2.1) provides S_{ab} once we know $\mathbf{\Omega}$ yielding

$$S_{ab} dx^a dx^b = \sigma_{\alpha\beta} dx^\alpha dx^\beta - 2\mathrm{g}_{\alpha A} dx^\alpha dx^A - \mathrm{g}_{AB} dx^A dx^B, \tag{8.1}$$

where $\sigma_{\alpha\beta}$ can be determined in terms of $\mathrm{g}_{\alpha\beta}$, g_{AB} and $\mathrm{g}_{\alpha A}$ using (2.1). Now, as we proved in proposition 3.1, the second equation of (3.1) is equivalent to $\pounds_{\vec{\boldsymbol{\xi}}} \mathbf{\Omega} = p(\alpha+\beta)\mathbf{\Omega}/2$ and this last equation takes the form

$$\pounds_{\vec{\boldsymbol{\xi}}} (\rho dx^1 \wedge \ldots \wedge dx^p) = (\pounds_{\vec{\boldsymbol{\xi}}} \rho) dx^1 \wedge \ldots \wedge dx^p + \rho d\xi^1 \wedge \ldots \wedge dx^p + \ldots +$$

$$+\rho dx^1 \wedge \ldots \wedge d\xi^p = (\pounds_{\vec{\boldsymbol{\xi}}} \rho + \partial_\alpha \xi^\alpha) dx^1 \wedge \ldots \wedge dx^p + \rho \partial_A \xi^1 dx^A \wedge dx^2 \wedge \ldots \wedge dx^p +$$

$$+ \ldots + \rho \partial_A \xi^p dx^1 \wedge \ldots \wedge dx^{p-1} \wedge dx^A = \frac{p}{2}(\alpha+\beta)\rho dx^1 \wedge \ldots \wedge dx^p,$$

from what we conclude

$$\frac{1}{\rho} \pounds_{\vec{\boldsymbol{\xi}}} \rho + \partial_\alpha \xi^\alpha = \frac{p}{2}(\alpha+\beta), \tag{8.2}$$

$$\partial_B \xi^\alpha = 0 \quad \Rightarrow \xi^\alpha = \xi^\alpha(x^\beta). \tag{8.3}$$

This is our first set of equations. The remaining ones come from the first equation of (3.1) which written in components looks like

$$\xi^a \partial_a \mathrm{g}_{mn} + \partial_m \xi^r \mathrm{g}_{rn} + \partial_n \xi^r \mathrm{g}_{mr} = a\mathrm{g}_{mn} + \beta S_{mn}. \tag{8.4}$$

If we spell out the different components, we get

$$\xi^a \partial_a \mathrm{g}_{\alpha\beta} + \partial_\alpha \xi^\gamma \mathrm{g}_{\gamma\beta} + \partial_\beta \xi^\gamma \mathrm{g}_{\alpha\gamma} + \partial_\alpha \xi^B \mathrm{g}_{B\beta} + \partial_\beta \xi^B \mathrm{g}_{\alpha B} = \alpha \mathrm{g}_{\alpha\beta} + \beta\sigma_{\alpha\beta} \tag{8.5}$$

$$\xi^\alpha \partial_\alpha \mathrm{g}_{AB} + \xi^C \partial_C \mathrm{g}_{AB} + \partial_A \xi^C \mathrm{g}_{CB} + \partial_B \xi^C \mathrm{g}_{AC} = (\alpha - \beta)\mathrm{g}_{AB} \tag{8.6}$$

$$\xi^\gamma \partial_\gamma \mathrm{g}_{A\alpha} + \xi^B \partial_B \mathrm{g}_{A\alpha} + \partial_A \xi^B \mathrm{g}_{B\alpha} + \partial_\alpha \xi^\gamma \mathrm{g}_{A\gamma} + \partial_\alpha \xi^B \mathrm{g}_{AB} = (\alpha - \beta)\mathrm{g}_{\alpha A}. \tag{8.7}$$

It is convenient to split the vector field $\vec{\boldsymbol{\xi}}$ in two parts

$$\vec{\boldsymbol{\xi}}_1 = \xi^\alpha \partial_\alpha, \quad \vec{\boldsymbol{\xi}}_2 = \xi^A \partial_A,$$

so that $\vec{\boldsymbol{\xi}} = \vec{\boldsymbol{\xi}}_1 + \vec{\boldsymbol{\xi}}_2$. Observe that $\vec{\boldsymbol{\xi}}_1$ is a genuine vector field in the distribution spanned by $\boldsymbol{\Omega}$. In terms of them the previous equations are rewritten as

$$(\pounds_{\vec{\boldsymbol{\xi}}_1} \mathrm{g})_{\alpha\beta} + (\pounds_{\vec{\boldsymbol{\xi}}_2} \mathrm{g})_{\alpha\beta} = \alpha \mathrm{g}_{\alpha\beta} + \beta\sigma_{\alpha\beta} \tag{8.8}$$

$$(\pounds_{\vec{\boldsymbol{\xi}}_1} \mathrm{g})_{AB} + (\pounds_{\vec{\boldsymbol{\xi}}_2} \mathrm{g})_{AB} = (\alpha - \beta)\mathrm{g}_{AB} \tag{8.9}$$

$$(\pounds_{\vec{\boldsymbol{\xi}}_1} \mathrm{g})_{A\alpha} + (\pounds_{\vec{\boldsymbol{\xi}}_2} \mathrm{g})_{A\alpha} = (\alpha - \beta)\mathrm{g}_{A\alpha}, \tag{8.10}$$

These are the most general set of equations for bi-conformal vector fields such that the projector P_{ab} can be regarded as conformally related to a true metric tensor defined on a submanifold of the total manifold $(V, \mathbf{g})$. The only nonvanishing components of this projector are

$$P_{\alpha\beta} = \frac{1}{2}(\sigma_{\alpha\beta} + \mathrm{g}_{\alpha\beta}).$$

An interesting case arises when $\mathrm{g}_{A\alpha} = 0$ or in other words the metric tensor g_{ab} is breakable in two parts as in (7.7), each of them depending on all the coordinates. From equation (8.10) (see (8.7)) we get that

$$\mathrm{g}_{A\alpha} = 0 \Rightarrow \partial_\alpha \xi^B = 0$$

so that $\vec{\boldsymbol{\xi}}_2$ depends only on the coordinates $\{x^A\}$ and it is a genuine vector field of the distribution spanned by $\{\partial/\partial x^A\}$. Furthermore $\sigma_{\alpha\beta}$ turns out to be $\mathrm{g}_{\alpha\beta}$ so (8.8) and (8.9) become

$$\pounds_{\vec{\boldsymbol{\xi}}_1} \mathrm{g}_{\alpha\beta} + \pounds_{\vec{\boldsymbol{\xi}}_2} \mathrm{g}_{\alpha\beta} = (\alpha + \beta)\mathrm{g}_{\alpha\beta} \tag{8.11}$$

$$\pounds_{\vec{\boldsymbol{\xi}}_2} \mathrm{g}_{AB} + \pounds_{\vec{\boldsymbol{\xi}}_1} \mathrm{g}_{AB} = (\alpha - \beta)\mathrm{g}_{AB}, \tag{8.12}$$

where the only nonvanishing components of the projectors P_{ab} and Π_{ab} are $P_{\alpha\beta} = \mathrm{g}_{\alpha\beta}$ and $\Pi_{AB} = \mathrm{g}_{AB}$. This couple of equations can be solved in an adapted coordinate system where $\vec{\boldsymbol{\xi}}_1$ and $\vec{\boldsymbol{\xi}}_2$ take the form $\vec{\boldsymbol{\xi}}_1 = \partial/\partial x^1$, $\vec{\boldsymbol{\xi}}_2 = \partial/\partial x^n$. The solution consists on a metric tensor such that the two pieces $\mathrm{g}_{\alpha\beta}$ and g_{AB} can be factored as

$$\mathrm{g}_{\alpha\beta} = f\, G_{\alpha\beta}, \quad \mathrm{g}_{AB} = h\, G_{AB}, \tag{8.13}$$

where $0 = \pounds_{\vec{\boldsymbol{\xi}}} G_{\alpha\beta} = \pounds_{\vec{\boldsymbol{\xi}}} G_{AB}$ and f, h are nonvanishing functions otherwise arbitrary. This can be compared with the form found in proposition 4.1 for a general metric in coordinates adapted to $\vec{\boldsymbol{\xi}}$.

Equation (8.13) is more general than (7.9) as $G_{\alpha\beta}$ and G_{AB} may depend on all coordinates: this is just a breakable space, not necessarily double-twisted. Nonetheless, the considerations pointed out after theorem 7.2 hold also here.

Interesting subcases of (8.13) using warped-product spacetimes were treated at the end of [23].

Example 3. In Lorentzian geometry, let us consider a bi-conformal vector field such that S_{ab} is the Lorentz tensor of a normalized timelike 1-form u_a. In physical terms this 1-form may represent the velocity vector field of a fluid, or the congruence associated to a set of observers, or a reference system, among others. In this case the explicit form of S_{ab} is found to be $S_{ab} = -\mathrm{g}_{ab} + 2u_a u_b$ and equations (3.1) (or (4.1)) become

$$\pounds_{\vec{\boldsymbol{\xi}}}(u_a u_b) = (\alpha+\beta)u_a u_b, \qquad \pounds_{\vec{\boldsymbol{\xi}}}(\mathrm{g}_{ab} - u_a u_b) = (\alpha-\beta)(\mathrm{g}_{ab} - u_a u_b), \tag{8.14}$$

from where

$$\pounds_{\vec{\boldsymbol{\xi}}}\, u_a = \frac{1}{2}(\alpha+\beta)u_a. \tag{8.15}$$

The tensor $h_{ab} \equiv \mathrm{g}_{ab} - u_a u_b$ is the orthogonal projector defined by the congruence whose tangent vector is u^a. It is worth remarking here that the case with α and β fixed constants is a symmetry already known in the literature as *kinematic self-similarity*. It was first introduced in [12] and was later studied specially in spherically-symmetric perfect-fluid spacetimes [38, 5, 15]. Kinematic self-similarity can be interpreted as saying that the flow generated by $\vec{\boldsymbol{\xi}}$ scales by unequal factors the timelike direction u^a and the spacelike directions contained in h_{ab} as opposed to *self-similarity* where $\vec{\boldsymbol{\xi}}$ is a homothetic Killing vector and the scaling takes place by the same factor in all the directions. As we see, (8.14) and (8.15) are a generalization of kinematic self-similarity — which is obviously included — such that the gauges are non-constant, so that the solutions for $\vec{\boldsymbol{\xi}}$ could be called "kinematic conformal vector".

Another interesting case of the above equations arises when $\xi^a = u^a$. In this case the acceleration 1-form a_a of the congruence defined by the vector field u^a is given by $\pounds_{\vec{\boldsymbol{u}}}\, u_a = a_a$ from what we deduce, using (8.15) and the orthogonality $a_a u^a = 0$, that $a_a = 0$ i.e. this is a geodesic congruence. Moreover, $\alpha = -\beta$ and the second equation of (8.14) becomes

$$\pounds_{\vec{\boldsymbol{u}}}\, h_{ab} = 2\alpha h_{ab}$$

which means that this is a shear-free congruence whose expansion is proportional to α (see e.g. [39] for definitions). In other words, every geodesic shear-free timelike congruence defines a bi-conformal vector field where the Lorentz tensor is $\mathbf{S} = \mathbf{T}\{\boldsymbol{u}\}$. These congruences have been extensively studied in General Relativity specially for perfect fluids whose velocity vector is u^a (see e.g. [18, 37, 31, 17]). It is known [37] that every such *perfect-fluid* congruence is either expansion-free or irrotational. In the former case, $\alpha = 0$ necessarily and the congruence is a *geodesic rigid motion* having the remarkable geometric property that it defines a *homogeneous* family of observers i.e. the average distance between neighbouring observers remains the same along the congruence. These congruences were first defined by Born [9] as the relativistic generalization of the rigid motions used in Newtonian mechanics and further studies of them can be found in [3, 4, 2, 32, 33, 42, 40] and references therein. The latter case can only be achieved if the perfect fluid defines a Robertson-Walker solution and again the fluid vector congruence defines a privileged observer.

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Appendix A: Number of independent constraints in equation (6.26)

In this appendix we will show how many linearly independent equations are contained in (6.26). This equation can be cast in the following equivalent form

$$\xi_c \nabla^c S_{ab} + M_{ab}{}^{qc}\Psi_{qc} = 0, \quad M_{ab}{}^{qc} = \delta^{[q}_{\ a} S^{c]}_{\ \ b} + \delta^{[q}_{\ b} S^{c]}_{\ \ a} = 2\delta^{[q}_{\ (a} S^{c]}_{\ \ b)} \quad \text{(A.1)}$$

We can regard this equation as an homogeneous system for the variables ξ_c and Ψ_{qc} where the indexes qc and ab are gathered in a block and thus considered as a single index as far as the calculations are concerned. Therefore we can rewrite (A.1) in an explicit matrix form

$$\begin{pmatrix} (\nabla \mathbf{S})_I{}^c & M_I{}^K \end{pmatrix} \begin{pmatrix} \xi_c \\ \Psi_K \end{pmatrix} = 0, \quad I = \{ab\}, \ K = \{qc\},$$

where $1 \leq K \leq n(n-1)/2$, $1 \leq I \leq n(n+1)/2$, $1 \leq c \leq n$. Hence the number of linearly independent equations present in (A.1) is given by the rank of the matrix of this homogeneous system. In order to calculate the rank of this matrix, we choose an orthonormal basis $\{\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_n\}$ of eigenvectors of $S^a_{\ b}$ and label them according to the following scheme

$$\mathbf{g}(\vec{\mathbf{e}}_\alpha, \vec{\mathbf{e}}_\alpha) = +1, \ \mathbf{g}(\vec{\mathbf{e}}_A, \vec{\mathbf{e}}_A) = -1, \quad 1 \leq \alpha \leq r, \ r+1 \leq A \leq n.$$

Thus in this basis $\mathbf{g} = \mathrm{diag}(\overbrace{1, \ldots, 1}^{r}, \overbrace{-1, \ldots, -1}^{n-r})$. In the remaining parts of this appendix we will follow the same convention used to label the elements of the orthonormal basis, namely, Greek indexes will run from 1 to r and capital Latin indexes from $r+1$ to n. In the above orthonormal basis the tensor $S^a_{\ b}$ looks like (see proposition 2.2)

$$S^a_{\ b} = \begin{pmatrix} \mathbb{I}_{p\times p} & \\ & -\mathbb{I}_{(n-p)\times(n-p)} \end{pmatrix},$$

where $\mathbb{I}_{m\times m}$ is the m-dimensional identity matrix. It is then clear that in this basis the only non vanishing components of $M_{ab}{}^{qc}$ are those with $a = q$, $b = c$ being the value of such components

$$M_{ab}{}^{ab} = \frac{1}{2}(\delta^a_{\ a} S^b_{\ b} - \delta^b_{\ b} S^a_{\ a}) = S^b_{\ b} - S^a_{\ a}, \ a < b \quad \text{(no summation)}.$$

All these components have different row and column indexes so the rank of $M_I{}^K$ is the total number of such components. Given the form of $S^a_{\ b}$ we find by a simple counting that such number turns out to be $p(n-p)$.

Let us now spell out the covariant derivative of S_{ab} in the above chosen orthonormal basis

$$\nabla_c S_{ab} = -\gamma^e_{ca} S_{eb} - \gamma^e_{cb} S_{ae} = -\gamma^b_{ca} S_{bb} - \gamma^a_{cb} S_{aa},$$

where γ^a_{bc} are the components of the connection in this basis and the last step follows using that S_{ab} is a diagonal tensor in this basis (there is no summation in the last step). In the above orthonormal frame, the connection components satisfy the symmetry properties

$$\gamma^a_{ba} = 0, \ \gamma^\alpha_{bB} = \gamma^B_{b\alpha}, \ \gamma^\alpha_{c\beta} + \gamma^\beta_{c\alpha} = 0, \ \gamma^A_{bB} + \gamma^B_{bA} = 0,$$

from what we conclude that the only nonvanishing components of $\nabla_c S_{ab}$ are (again there is no summation in the repeated indexes)

$$\nabla_c S_{B\alpha} = -\gamma^{\alpha}_{cB}(S_{\alpha\alpha}+S_{BB}),\ \nabla_c S_{AB} = -\gamma^{B}_{cA}(S_{BB}-S_{AA}),\ \nabla_c S_{\alpha\beta} = -\gamma^{\beta}_{c\alpha}(S_{\beta\beta}-S_{\alpha\alpha}).$$

Comparing these equations with the above formulae for ${M_I}^K$ we infer that ${M_I}^K = 0$ $\forall K$ implies that ${(\nabla\mathbf{S})_I}^c$ vanishes for every value of the index c so the rank of the matrix system is just the rank of ${M_I}^K$ calculated above.

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